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Characteristic Classes and Integrable Systems for Simple Lie Groups.

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Abstract

This paper is a continuation of our previous paper [7]. For simple complex Lie groups with non-trivial center i.e. classical simply-connected groups, E_6 and E_7 we consider elliptic Modified Calogero-Moser systems corresponding to the Higgs bundles with an arbitrary characteristic class. These systems are generalization of the classical Calogero-Moser (CM) systems related to a simple Lie groups and contain CM systems related to some (unbroken) subalgebras. For all algebras we construct a special basis, corresponding to non-trivial characteristic classes, the explicit forms of Lax operators and Hamiltonians.

Contents

1 Introduction

This paper is a continuation of our previous paper [7]. Here we consider concrete implementation of the generic formulae for all simple groups with a non-trivial center. In particular, we find the structure of the unbroken Lie subalgebras $\tilde{\mathfrak{g}}_0$ (see Table I in [7]). We refer the formulae from [7] with a number I. The information about roots, weights and so on was taken from [1, 4].

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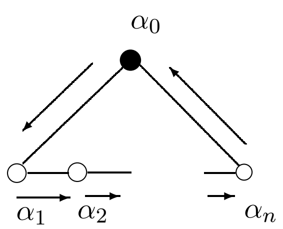
2 $\mathrm{SL}(N,\mathbb{C})$ - the root system A_{N-1}

Roots and weights.

For the A_n, $(n = N-1)$ root system we have two groups $\bar{G} = \mathrm{SL}(N,\mathbb{C})$, $G^{ad} = \mathrm{PSL}(N,\mathbb{C})$. Choose the Cartan subalgebra $\mathfrak{H} \subset \mathfrak{g}$ as an subalgebra of traceless diagonal matrices. Then $\mathfrak{H}$ can be identify with the hyperplane in $\mathbb{C}^N$ $\mathfrak{H} = \{\mathbf{x} = (x_1,\ldots,x_N) \in \mathbb{C}^N \mid \sum_{j=1}^N x_j = 0\}$. The simple roots $\Pi = \{\alpha_k\} = \{\alpha_1 = e_1 - e_2 \ldots, \alpha_{N-1} = e_{N-1} - e_N\}$ form a basis in the dual space $\mathfrak{H}^*$. Here $\{e_j\}$ $j = 1,\ldots,N$ is a canonical basis in $\mathbb{C}^N$. The vectors e_j generate the set of roots $R = \{(e_j - e_k),\ j \neq k\}$ of type A_{N-1}. The minimal root is

$$\alpha_0 = -\sum_{\alpha\in\Pi} \alpha_k = e_N - e_1\,. \tag{2.1}$$

It defines the extended Dynkin diagram



`Fig.1`: A_n and action of λ_{N-1}

Since the half-sum of positive roots is $\rho = \frac{1}{2}(N-1, N-3, \ldots, 3-N, 1-N)$ and $h = N$, $\kappa = \frac{\rho}{h} = \frac{1}{2N}(N-1, N-3, \ldots, 3-N, 1-N)$. For $\alpha = e_k - e_{k+a}$ the level (A.3.I) $f_\alpha = a$ and

$$\langle\kappa,\alpha\rangle = a/N \tag{2.2}$$

(see (A.14.I)).

We identify $\mathfrak{H}^*$ and $\mathfrak{H}$ by means of the standard metric on $\mathbb{C}^N$. Then the coroot system coincides with R, and the coroot lattice $Q^\vee$ coincides with Q

$$Q = \{\sum m_j e_j \,|\, m_j \in \mathbb{Z}\,,\ \sum m_j = 0\}\,. \tag{2.3}$$

The fundamental weights ϖ_k, $(k = 1,\dots,N-1)$, dual to the basis of simple roots $\Pi^\vee \sim \Pi$ $(\varpi_k(\alpha_k^\vee) = \delta_{kj})$, are

$$\varpi_j = e_1 + \ldots + e_j - \frac{j}{N}\sum_{l=1}^{N} e_l\,, \qquad \left\{ \begin{array}{l} \varpi_1 = (\frac{N-1}{N}, -\frac{1}{N}, \dots, -\frac{1}{N}) \\ \varpi_2 = (\frac{N-2}{N}, \frac{N-2}{N}, \dots, -\frac{2}{N}) \\ \dots \\ \varpi_{N-1} = (\frac{1}{N}, \frac{1}{N}, \dots, \frac{1-N}{N})\,. \end{array}\right. \tag{2.4}$$

In the basis of simple roots the fundamental weights are

$$\varpi_k = \frac{1}{N}[(N-k)\alpha_1 + 2(N-k)\alpha_2 + \ldots (k-1)(N-k)\alpha_{k-1}$$

$$+k(N-k)\alpha_k + k(N-k-1)\alpha_{k+1} + \ldots + k\alpha_{N-1}]\,.$$

Similar to the roots and coroots we identify the fundamental weights and the fundamental coweights. They generate the weight (coweight) lattice

$$P \subset \mathfrak{H}\,,\ P = \{\sum_l n_l\varpi_l \,|\, n_l \in \mathbb{Z}\}\,,\ \text{ or } P = \{\sum_{j=1}^{N} m_j e_j\,,\ m_j \in \frac{1}{N}\mathbb{Z}\,,\ m_j - m_k \in \mathbb{Z}\}\,. \tag{2.5}$$

or

$$P = Q + \mathbb{Z}\varpi_{N-1}\,,\ \ (\varpi_{N-1} = -e_N + \frac{1}{N}\sum_{j=1}^{N} e_j)\,. \tag{2.6}$$

Transition matrices.

The factor-group $P^\vee/Q^\vee \sim P/Q$ is isomorphic to the center $\mathcal{Z}(\mathrm{SL}(N,\mathbb{C})) \sim \mu_N$. It is generated by $\zeta = \exp 2\pi i\varpi_{N-1}$. Following Proposition 3.1.I define λ_{N-1} and its action on the extended Dynkin diagram. Consider the fundamental alcove (A.17.I). It follows from (2.1) that

$$C_{alc} = \{0, \varpi_1, \dots, \varpi_{N-1}\}\,.$$

Thus, any fundamental weight generates a nontrivial Λ. For $\xi = \varpi_{N-1}$ in (3.10.I) we find the corresponding transformation

$$\lambda_{N-1}\ :\ e_j \to e_{j+1}\,,\ (\alpha_k \to \alpha_{k+1})\,. \tag{2.7}$$

The action of λ_{N-1} on Π^{ext} is presented on Fig.1. It means that in the canonical basis in $\mathbb{C}^N$ λ_{N-1} acts as the permutation matrix

$$\Lambda = c\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}\,, \qquad c = \left\{ \begin{array}{cc} 1\,, & N = 2m+1 \\ \exp\frac{\pi i}{2m}\,, & N = 2m \end{array}\right.\,. \tag{2.8}$$

Evidently, that $\Lambda \in \mathrm{SL}(N,\mathbb{C})$ and $\Lambda^N \sim Id$. On the other hand

$$\mathcal{Q} = \mathbf{e}\,(\kappa) = \mathrm{diag}(\mathbf{e}\left(\frac{N-1}{N}\right), \mathbf{e}\left(\frac{N-3}{N}\right), \dots, \mathbf{e}\left(\frac{1-N}{N}\right)), \quad (\mathbf{e}(x) = \exp{(2\pi i x)})\,. \tag{2.9}$$

Thus,

$$[\Lambda, \mathcal{Q}] = \zeta\,, \quad \zeta = \omega \cdot Id\,, \; \omega = \mathbf{e}\,(1/N)\,,$$

and ζ is an obstruction to lift a $\mathrm{PSL}(N,\mathbb{C})$-bundle to a $\mathrm{SL}(N,\mathbb{C})$-bundle. Since $Ker\,(\lambda_{N-1} - 1)|_{\mathfrak{H}} = 0\ \tilde{\mathfrak{H}} =$ (see Proposition 3.1.I) the moduli space is empty set (see (3.29.I) and (3.32.I)).

Consider another extreme case when ξ lies in the root lattice Q. Then $\Lambda = Id$ and $\mathfrak{H} \in Ker\Lambda$. Since $[\Lambda, \mathcal{Q}] = Id$, the bundles have a trivial characteristic class.

Consider an intermediate case and assume that N has nontrivial divisors $N = pl$, $(l \neq 1, N)$. It can be found that

$$\varpi_j \,\to\, \lambda_j \,:\, e_k \to e_{k+N-j}\,. \tag{2.10}$$

Then there exists a sublattice $P_l = Q + \mathbb{Z}\varpi_{N-p}\,,\ Q \subset P_l \subset P\,,$ such that $P/P_l \sim \mu_l$. It follows from (2.10) that $\lambda_{N-p} \,:\, e_j \to e_{j+p}$. Therefore, Λ_{N-p} is equal the p-th degree of (2.8) $(\Lambda^l_{N-p} = Id)$ For $\zeta_p = \mathbf{e}(\varpi_{N-p}) = \omega^p Id\,,\ (\omega^N = 1)$ (3.3.I) takes the form

$$[\Lambda_{N-p}, \mathcal{Q}] = \omega^p \cdot Id_N\,, \tag{2.11}$$

where $\mathcal{Q}$ is (2.9).

Define the factor group G_l

$$1 \to \mu_l \to \mathrm{SL}(N,\mathbb{C}) \to G_l \to 1\,. \tag{2.12}$$

It has the center $\mathcal{Z}(G_l) \sim \mu_p = \mathbb{Z}/p\mathbb{Z}$. Therefore,

$$1 \to \mu_p \to G_l \to \mathrm{PSL}(N,\mathbb{C}) \to 1\,.$$

The sublattice P_l is isomorphic the group of cocharcters $t(G_l)$ (A.43.I)0

For the dual group ${}^LG_l = G_p$ we have the similar exact sequences, where the role of l and p are changed

$$1 \to \mu_p \to \mathrm{SL}(N,\mathbb{C}) \to G_p \to 1\,,$$
$$1 \to \mu_l \to G_p \to \mathrm{PSL}(N,\mathbb{C}) \to 1\,. \tag{2.13}$$

It follows from (2.12) and (2.13) that the cocycle $H^2(\Sigma_\tau, \mu_l)$ is obstruction to lift G_l-bundles to $\mathrm{SL}(N,\mathbb{C})$ bundles or to lift $\mathrm{PSL}(N,\mathbb{C})$-bundles to $G_p =^L G_l$-bundles.

Bases

Sin-basis.

Consider the case $\zeta = \exp(2\pi i \varpi_{N-1})$. Therefore Λ is (2.8) and all it orbits in R have the same length N. The number of orbits is $N-1$. In other words $\sharp R = N(N-1)$. Let us take the root subspaces $E_{1,a}$ $(a \neq 1)$ as representatives of orbits in the space $\mathfrak{L}$. Then the basis (5.4.I) in $\mathfrak{L}$ is

$$\mathfrak{t}^c_a = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} \omega^{mc} E_{1+m,a+m}\,,\ c = (0,\dots,N-1)\,,\ \ \omega^N = 1\,.$$

In particular,

$$\mathfrak{t}_1^c = \frac{1}{\sqrt{N}}\begin{pmatrix} 0 & \omega^c & 0 & \dots & 0 \\ 0 & 0 & \omega^{2c} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & \omega^{(N-1)c} \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

There is only one orbit in the Cartan subalgebra $\mathfrak{H}$ under the action of λ_{N-1}

$$e_1 \to e_2 \to e_3, \to \dots, e_N, .$$

For the A_{N-1} roots it is convenient to pass to the canonical over-complete basis $(e_1, e_2, \dots, e_N)$. Then the basis (5.33.I) on $\mathfrak{H}$ is

$$\mathfrak{h}^c = \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1}\omega^{mc}e_{m+1} = \frac{1}{\sqrt{N}}\mathrm{diag}(1, \omega^c, \dots, \omega^{(N-1)c}),\ (c = 1, \dots, N-1).$$

Essentially, $(\mathfrak{t}_a^c, \mathfrak{h}^c)$ form a basis of the sin-algebra [2].

If N is a primitive number then the center $\mathcal{Z}(\mathrm{SL}(N,\mathbb{C})) = \mu_N$ has not nontrivial subgroups. Therefore, one can put in (3.9.I) $\xi \in P$. This case leads to the sin-basis. Another options is $\xi \in Q$. In this case $\lambda = Id$ and we come to the Chevalley basis (see Remark 5.1.I).

Generalized Sin-Basis in $\mathrm{SL}(N,\mathbb{C})$**.**

Let $N = pl$ and $\xi = \varpi_{N-p}$ generates $\Lambda_{N-p} = \Lambda^p$. All orbits of μ_l have the length l. In the space of bases $\mathcal{E} = \{E_{jk}\,,\ j \neq k$ there are $p(N-1)$ orbits passing through the matrices

$$E_{s,s+a}\,,\quad (s = 1, \dots, p\,,\ a = 1, \dots, N\,,\ (mod\, N))\,. \tag{2.14}$$

The off-diagonal basis is

$$\mathfrak{t}_{s,k}^c = \frac{1}{\sqrt{l}}\sum_{m=0}^{l-1}\omega^{mpc}E_{s+mp,s+k+mp}\,,\quad \omega^N = 1\,, \tag{2.15}$$

$$c = 0, \dots, l-1\,,\ s = 1, \dots, p\,,\ k = 1, \dots, N-1\,, (mod\, N))\,.$$

The pairing in this basis (5.8.I) assumes the form

$$(\mathfrak{t}_{s_1,k}^a, \mathfrak{t}_{s_2,j}^b) = \delta_{k,-j}\delta^{(a,-b\,, mod\, l)}\delta_{(s_1+k,s_2+rp)}\omega^{rpb}\,,\quad (s_1 + k - s_2 = rp)\,. \tag{2.16}$$

Let $(e_1, \dots, e_N)$ be the canonical basis in $\mathfrak{H}$. There are p orbits of length l

$$\begin{aligned} &\tilde{e}_1 = e_1 + e_{p+1} + \dots + e_{(l-1)p+1} \\ &\tilde{e}_2 = e_2 + e_{p+2} + \dots + e_{(l-1)p+2} \\ &\dots \\ &\tilde{e}_p = e_p + e_{2p} + \dots + e_{lp}\,. \end{aligned}$$

Let $\tilde{\mathfrak{H}}_0 \subset \mathfrak{H}$ be a Cartan subalgebra

$$\tilde{\mathfrak{H}} = \{\tilde{\mathbf{u}} = \sum_{j=1}^p u_j\tilde{e}_j \mid \sum_{j=1}^p u_j = 0\}\,.$$

with the basis of simple coroots

$$\tilde{\Pi}^{\vee} = \{\tilde{\alpha}^{\vee}_k = \tilde{e}_k - \tilde{e}_{k+1}\}\,. \tag{2.17}$$

The simple roots

$$\tilde{\Pi} = \{\tilde{\alpha}_k = \frac{1}{l}(\tilde{e}_k - \tilde{e}_{k+1})\}$$

along with $\tilde{\Pi}^{\vee}$ generate A_{p-1} type Cartan matrix $a_{jk} = \langle \tilde{\alpha}^{\vee}_j, \tilde{\alpha}_k \rangle$. The simple coroots and the subspaces

$$\tilde{E}_{i,a} = \sum_{m=0}^{l-1} E_{i+mp,i+a+mp}\,, \quad (1 \le i \le p\,,\ a = \pm 1, \pm 2, \ldots \pm (p-1)) \tag{2.18}$$

form the Chevalley basis in the invariant subalgebra $\tilde{\mathfrak{g}}_0 = \mathrm{sl}(p,\mathbb{C})$ (see Proposition 5.1).

It follows from (2.15) that the basis (5.29.I) in the space V (5.30.I) takes the form

$$V = \{\mathfrak{t}^0_{s,a} = \frac{1}{\sqrt{l}} \sum_{m=0}^{l-1} E_{s+mp,s+a+mp}\,,\ (1 \le s \le p\,,\ p \le a \le N)\}\,. \tag{2.19}$$

Together with $\tilde{\mathfrak{g}}_0$, V forms the invariant subalgebra

$$\mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 = \underbrace{\mathrm{sl}(p,\mathbb{C}) \oplus \ldots \oplus \mathrm{sl}(p,\mathbb{C})}_{l} \oplus \underbrace{(\mathbb{C} \oplus \ldots \oplus \mathbb{C})}_{l-1}$$

Its structure is obtained from Π^{ext} by dropping l roots $(\alpha_p,\ \ldots, \alpha_{(l-1)p}, \alpha_0)$. This procedure defines an automorphism of $\mathfrak{g}_{A_{N-1}}$ of order $l = N/p$.

The Killing form in $\tilde{\mathfrak{g}}_0$ is

$$(\tilde{e}_k, \tilde{e}_j) = l\delta_{kj}\,, \quad (\tilde{E}_{i,a}, \tilde{E}_{j,b}) = l\delta_{a,-b}\delta_{i+a,j\,,\,(mod\,p)} \tag{2.20}$$

and commutation relations

$$[\tilde{e}_k, \tilde{E}_{i,a}] = (\delta_{i,k} - \delta_{i-k+a,0\,mod\,p})\tilde{E}_{i,a}\,,$$

$$[\tilde{E}_{i,a}, \tilde{E}_{j,b}] = \delta_{i+a,j\,mod\,p}\tilde{E}_{i,a+b} - \delta_{i,j+b\,mod\,p}\tilde{E}_{j,a+b}\,,\ (b \ne p-a)\,,$$

$$[\tilde{E}_{i,a}, \tilde{E}_{i+a,p-a}] = \tilde{e}_i - \tilde{e}_j\,.$$

The basis in $\mathfrak{H}$ is generated by $\tilde{e}_j$ and by

$$\mathfrak{h}^c_j = \frac{1}{\sqrt{l}} \sum_{m=0}^{l-1} \omega^{mpc} e_{j+mp}\,, \quad c = 1, \ldots, l-1\,, \quad j = 1, \ldots, p \tag{2.21}$$

with the pairing

$$(\mathfrak{h}^{c_1}_j, \mathfrak{h}^{c_2}_k) = \delta_{j,k}\delta^{c_1,-c_2}\,. \tag{2.22}$$

The subspace $\mathfrak{g}_c$ in (5.1.I) is formed by the basis $\{\mathfrak{h}^c_j$ (2.21)$\,,\ \ \mathfrak{t}^c_{s,a}$ (2.15)$\,,\ c \ne 0\}$.

The general commutation relations in this basis

$$[\mathfrak{t}^{c_1}_{i,a}, \mathfrak{t}^{c_2}_{j,b}] = \frac{1}{\sqrt{l}}\left(\omega^{(i-j+a)c_2}\delta_{j,i+a\,(mod\,p)}\mathfrak{t}^{c_1+c_2}_{i,a+b} - \omega^{(j-i+b)c_1}\delta_{i,j+b\,(mod\,p)}\mathfrak{t}^{c_1+c_2}_{j,a+b}\right)\,, \tag{2.23}$$

$$[\tilde{e}_i, \mathfrak{t}^c_{i,a}] = \delta_{j,i}\mathfrak{t}^c_{i,a} - \delta_{i,j+b\,(mod\,p)}\mathfrak{t}^c_{j,b}\,, \tag{2.24}$$

$$[\mathfrak{h}^{c_1}_i, \mathfrak{t}^{c_2}_{j,b}] = \frac{1}{\sqrt{l}}\left(\omega^{(i-j)c_2}\delta_{j,i}\mathfrak{t}^{c_1+c_2}_{i,b} - \omega^{(j-i+b)c_1}\delta_{i,j+b\,(mod\,p)}\mathfrak{t}^{c_1+c_2}_{j,b}\right)\,, \tag{2.25}$$

The Lax operators and the Hamiltonians.

Trivial bundles

For trivial bundles $\xi \in Q$, $\Lambda = Id$ and $\mathfrak{H} \subset Ker\Lambda$. One can consider trivial SL$(N,\mathbb{C})$ or PSL$(N,\mathbb{C})$ bundles. They differ by their moduli spaces (3.21.I) or (3.22.I). At the case at hand they assume the following form. Let

$$C^{+} = \{\tilde{\mathbf{u}} \in \mathbb{C}^N \,|\, u_1 \geq u_2 \geq \ldots \geq u_N\}$$

be a positive Weyl chamber (A.8.I). Then (3.21.I) and (3.22.I) take the form

$$C^{SL} = \{\mathbf{u} \in C^{+} \,|\, u_j \sim u_j + \tau n_j + m_j\,,\ n_j, m_j \in \mathbb{Z}\,,\ \sum_j n_j = \sum_j m_j = 0\}\,. \tag{2.26}$$

$$C^{PSL} = \{\mathbf{u} \in C^{SL} \,|\, n_j, m_j \in \frac{1}{N}\mathbb{Z}\,,\ n_j - n_k \in \mathbb{Z}\,,\ m_j - m_k \in \mathbb{Z}\ \}\,. \tag{2.27}$$

The Lax operator is the SL(N), (or PSL(N)) and the Hamiltonian CM system are well-known [3, 9, 12]

$$L^{CM}_{SL(N)} = \sum_{j=1}^{N}(v_j + S_j)e_j + \sum_{j\neq k} S_{jk}\phi(u_k - u_j, z)E_{jk}\,,$$

$$H^{CM}_{SL(N)} = \frac{1}{2}\sum_{j=1}^{N} v_j^2 - \sum_{j\neq k} S_{jk}S_{kj}E_2(u_j - u_k)\,.$$

Nontrivial bundles

Define a moduli space of bundles with characteristic class ω^p (2.11). Evidently, $\tilde{\mathfrak{H}} = Ker(\Lambda_{N-p} - 1)|_{\tilde{\mathfrak{H}}_0}$. Let $Q_l \subset Q$ be an invariant coroot lattice in $\tilde{\mathfrak{H}}_0 \subset \mathbf{sl}(p,\mathbb{C})$

$$Q_l^{\vee} = \{\gamma = \sum_{j=1}^{p} m_j\tilde{e}_j\,,\ \ m_j \in \mathbb{Z}\,,\ \sum_{j=1}^{p} m_j = 0\}\,,$$

generated by the simple coroots (2.17). It is an invariant sublattice of $Q^{\vee}$ with respect to the Λ_{N-p} action.

The fundamental coweights $(\langle\tilde{\varpi}_j^{\vee}, \tilde{\alpha}_k\rangle = \delta_{jk})$

$$\begin{array}{l}
\tilde{\varpi}_1^{\vee} = \frac{p-1}{p}\tilde{e}_1 - \frac{1}{p}\tilde{e}_2 - \ldots \frac{1}{p}\tilde{e}_p\,,\\
\tilde{\varpi}_2^{\vee} = \frac{p-2}{p}\tilde{e}_1 + \frac{p-2}{p}\tilde{e}_2 - \ldots - \frac{2}{p}\tilde{e}_p\,,\\
\ldots\ldots\\
\tilde{\varpi}_p^{\vee} = \frac{1}{p}\tilde{e}_1 + \frac{1}{p}\tilde{e}_2 + \ldots \frac{1-p}{p}\tilde{e}_p
\end{array}$$

form a basis in the coweight lattice

$$P_l^{\vee} = \{\gamma = \sum_{j=1}^{p} n_j\tilde{e}_j\,,\ \ n_j \in \frac{1}{p}\mathbb{Z}\,,\ \ \sum_{j=1}^{p} n_j = 0\,,\ \ n_j - n_k \in \mathbb{Z}\}\,.$$

It is an invariant sublattice of $P^{\vee}$.

Let $\tilde{W}$ is a permutation group of $\tilde{e}_1,\dots,\tilde{e}_p$. It is a Weyl group of $\tilde{R}(\tilde{\Pi})$. Define the semidirect products (3.28.I), (3.31.I)

$$\tilde{W}_{BS}=\tilde{W}\ltimes(\tau Q_l^\vee+Q_l^\vee)\,,\quad \tilde{W}_{BS}^{ad}=\tilde{W}\ltimes(\tau P_l^\vee+P_l^\vee)\,.$$

The moduli space is defined as in (3.29.I), (3.31.I)

$$C^{(l)\,sc}=\tilde{\mathfrak{H}}/\tilde{W}_{BS}\,,\quad C^{(l)\,ad}=\tilde{\mathfrak{H}}/\tilde{W}_{BS}^{ad}\,. \tag{2.28}$$

Thus, for

$$C^{(l)\,sc}\;:\;u_j\sim u_j+\tau m_j+n_j\,,\;n_j,m_j\in\mathbb{Z}\,,\;\sum_{j=1}^p n_j=\sum_{j=1}^p m_j=0\,,$$

and for

$$C^{(l)\,ad}\;:\;u_j\sim u_j+\tau m_j+n_j\,,\qquad n_j,m_j\in\frac{1}{p}\mathbb{Z}\,,$$

$$\sum_{j=1}^p n_j=\sum_{j=1}^p m_j=0\,,\quad n_j-n_k\in\mathbb{Z}\,,\quad m_j-m_k\in\mathbb{Z}\,.$$

Define $\tilde{\mathbf{u}}\in\mathfrak{H}$ such that

$$u_i=u_s \text{ if } i=s+mp\,. \tag{2.29}$$

It means that $\tilde{\mathbf{u}}\in\tilde{\mathfrak{H}}_0$, and $\tilde{\mathbf{u}}=\sum_{i=1}^p u_i\tilde{e}_i$. The Lax operator $L=\tilde{L}_0+L_0'+\sum_{a=1}^{l-1}L_a$ (6.16.I), (6.17.I), (6.18.I) takes the form, (see (2.2))

$$L_a(z)=\sum_{j=1}^p S_j^a\phi(\frac{a}{l},z)\mathfrak{h}_j^a+\sum_{i,k=1}^N S_{i,k}^a\mathbf{e}(zk/N)\phi(u_{i+k}-u_i+\tau k/N+\frac{a}{l},z)\mathfrak{t}_{i,k}^a\,,$$

$$\tilde{L}_0(z)=\sum_{i=1}^p(v_i+S_iE_1(z))\tilde{e}_i+\sum_{i,k}\tilde{S}_{i,k}\mathbf{e}(zk/N)\phi(u_{i+k}-u_i+\tau k/N,z)\tilde{E}_{i,k}\,,$$

$$L_0'(z)=\sum_{i,k}S_{i,k}'\mathbf{e}(zk/N)\phi(u_{i+k}-u_i+\tau k/N,z)\mathfrak{t}_{i,k}^0\,. \tag{2.30}$$

To come to an integrable system impose the moment constraints $\tilde{S}_i^{\tilde{\mathfrak{H}}}=0$ $(i=1,\dots,p)$ and the gauge fixing constraints. The calculation of the quadratic Hamiltonian is based on the pairing relations (2.16), (2.22), (2.20). Using (5.1.I) and (2.16) we come to the Hamiltonian

$$H=\tilde{H}_0+H_0'+\sum_{k=1}^{[l/2]}H_k\,.$$

Here $\tilde{H}_0$ is the sl(p) Calogero-Moser Hamiltonian

$$\tilde{H}_0=l\Big(\frac{1}{2}\sum_{i=1}^p v_i^2-\sum_{i=1}^p\sum_{k=1}^{p-1}\tilde{S}_{i,k}\tilde{S}_{i+k,-k}E_2(u_i-u_{i+k})\Big)\,.$$

For H_a and H_0' we have

$$H_0'=-\frac{1}{2}\sum_{i=1}^p\sum_{k=p+1}^N S_{i,k}^0S_{i+k+rp,-k}^0E_2(u_{i+k}-u_i+\tau k/N)\,,$$

$$H_a = -\frac{1}{2}\sum_{s=1}^{p}\Big(S_s^a S_s^{-a}E_2(a/l) + \sum_{k=1}^{N}\omega^{-aip}S_{s,k}^a S_{s+k+rp,-k}^{-a}E_2(u_{s+r}-u_s+a/l+\tau k/N)\Big)\,.$$

These Hamiltonians and the corresponding Lax operators were obtained in [8]. The basis, used there is not the GS basis. The corresponding Hamiltonians describe p interacting EA tops with inertia tensors depending on $\tilde{\mathbf{u}}$. This interpretation is specific for SL$(N,\mathbb{C})$ $(N=pl)$. In particular, if $\xi=\varpi_{N-1}$, $\tilde{\mathfrak{H}}_0=\emptyset$ and we deal with the sin-basis. The corresponding bundle has no moduli ($p=1$, $l=N$). The invariant Hamiltonian $\tilde{H}_0=0$ and H_0', H_a describe the so-called the Elliptic top [6, 11].

3 SO$(2n+1)$, Spin$(2n+1)$, B_n root system.

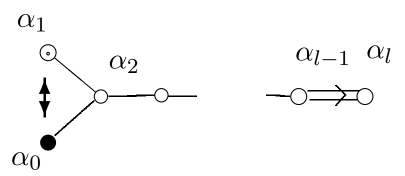
Roots and weights.

For the Lie algebra B_n the universal covering group $\bar{G}$ is Spin$(2n+1)$ and G^{ad} =SO$(2n+1)$. The simple roots of B_n are

$$\Pi_{B_n}=\{\alpha_j=e_j-e_{j+1}\,,\ j=1,\dots,n-1\,,\ \alpha_n=e_n\}\,.,\tag{3.1}$$

and the minimal root is

$$\alpha_0=-e_1-e_2=-(\alpha_1+2\alpha_2+\ldots+2\alpha_n)\,.\tag{3.2}$$



`Fig.2:` B_n and action of λ_1

The root system R_{B_n} contains $2n$ short roots $\pm e_j$ $(j=1,\dots n)$ and $2n(n-1)$ long roots $(e_j\pm e_k)$ $(j\neq k)$. The positive roots are

$$R^+=\{(e_j\pm e_k)\,,\ (j>k)\,,\quad e_j\,,\quad (j,k=1,\dots,n)\}\,.\tag{3.3}$$

The Weyl group of R_{B_n} is the semi-direct product of the permutation group S_n acting on e_j and the signs changing $e_j\to -e_j$

$$W_{B_n}=S_n\ltimes(\sum\mu_2)\,.\tag{3.4}$$

The coroot basis H_α in $\mathfrak{H}_{B_n}$ is formed by simple roots of the dual system

$$\Pi_{B_n}^\vee=\Pi_{C_n}=\{\alpha_j=e_j-e_{j+1}\,,\ j=1,\dots,n-1\,,\quad \alpha_n=2e_n\}\,.\tag{3.5}$$

$\Pi_{B_n}^\vee$ generates the coroot lattice

$$Q_{B_n}^\vee=Q_{C_n}=\{\gamma=\sum_{j=1}^n m_je_j\,|\,\sum_{j=1}^n m_j\text{ is even}\}\,.\tag{3.6}$$

The half-sum of positive coroots is $\rho^\vee_{B_n} = ne_2 + (n-1)e_2 + \ldots + 2e_{n-1} + e_n$, and since $h = 2n$

$$\kappa_{B_n} = \rho/h = \frac{1}{2}e_1 + (\frac{1}{2} - \frac{1}{2n})e_2 + \ldots + \frac{1}{n}e_{n-1} + \frac{1}{2n}e_n)\,. \tag{3.7}$$

The dual to Π_{B_n} the system of the fundamental coweights takes the form

$$\{\varpi^\vee_j = e_1 + e_2 + \ldots + e_j\,,\ j = 1, \ldots, n\}\,.$$

It generates the coweight lattice

$$P^\vee_{B_n} = P_{C_n} = \{\gamma = \sum_{j=1}^n m_j e_j \,|\, m_j \in \mathbb{Z}\}\,. \tag{3.8}$$

The factor group $P^\vee_{B_n}/Q^\vee_{B_n} \sim \mu_2$ is isomorphic to the center of $Spin(2n+1)$. The center is generated by the coweight $\varpi^\vee_1$ ($\zeta = \mathbf{e}(\varpi^\vee_1)$). It follows from (A.17.I) and (3.2) that the fundamental alcove has the vertices

$$C_{alc} = (0, \varpi^\vee_1, \frac{1}{2}\varpi^\vee_2, \ldots, \frac{1}{2}\varpi^\vee_n)\,.$$

We use this expression to find λ_1 corresponding to $\xi = \varpi^\vee_1$: (3.10.I)

$$\lambda_1 : (e_1 \to -e_1\,, e_j \to e_j\,,\ (j = 2, \ldots, n))\,. \tag{3.9}$$

It acts on $\Pi^{ext}_{B_n}$ as

$$\lambda_1 : (\alpha_1 \to \alpha_0\,,\ \alpha_j \to \alpha_j\,,\ (j = 2, \ldots, n))\,.$$

It is clear that $\tilde{\mathfrak{H}} = \mathfrak{H}_{B_{n-1}}$ and the moduli vector is $\tilde{\mathbf{u}} = (u_2, \ldots, u_n)$.

Bases.

The Chevalley basis

The Chevalley basis in $\mathfrak{g}_{B_n}$ is generated by the simple coroots $\Pi^\vee_{C_n} = \Pi_{B_n}$ (3.1) in $\mathfrak{H}_{B_n}$ and by the root subspaces. To describe the Chevalley basis for the classical groups we use their fundamental representations. For $\mathfrak{g}_{B_n}$ it is a fundamental representation π_1 corresponding to the weight $\varpi_1 = e_1$. It has dimension $2n+1$. $\mathfrak{g}_{B_n}$ becomes the Lie algebra of matrices of order $2n+1$ satisfying the constraints

$$Zq + qZ^T = 0\,,$$

where q in the bilinear form

$$q = \begin{pmatrix} 0 & 0 & J \\ 0 & 1 & 0 \\ J & 0 & 0 \end{pmatrix}\,,$$

and J is an n-th order anti-diagonal matrix

$$J = \begin{pmatrix} & & 1 \\ & 1 & \\ & & \\ 1 & & \end{pmatrix} \tag{3.10}$$

Then Z takes the form

$$Z = \begin{pmatrix} A & \alpha & B \\ \tilde{\beta}^T & 0 & -\tilde{\alpha}^T \\ C & -\beta & -\tilde{A} \end{pmatrix} , \quad B = \tilde{B}\,,\; C = \tilde{C}\,,\;\; \tilde{X} = JX^TJ\,, \tag{3.11}$$

$$\alpha = (\alpha_1, \dots, \alpha_n)\,, \quad \beta = (\beta_1, \dots, \beta_n)\,, \qquad \tilde{\alpha} = \alpha J\,.$$

The basis in $\mathfrak{H}_{B_n}$ is generated by the canonical basis $(e_1, \dots, e_n)$. Then the canonical basis in $\pi_1(\mathfrak{H})$ is

$$\mathrm{diag}(e_1, \dots, e_n, 0, -e_n, \dots, -e_1)\,.$$

The root subspaces in π_1 are

$$(e_j - e_k)\,,\; j \neq k \to \mathfrak{G}^-_{jk} = (E_{j,k} \in A - E_{2n+2-k,2n+2-j} \in A\,, \tilde{A})\,, \tag{3.12}$$

$$j < k \sim \text{(positive roots)}\,, \quad j > k \sim \text{(negative roots)}\,,$$

$$(e_j + e_k)\,, \to \mathfrak{G}^+_{jk} = (E_{j,k+n+1} - E_{n+1-k,2n+2-j}) \in B\,, C\,,$$

$$e_j\,, \to \mathfrak{G}^+_j = \sqrt{2}(E_{j,n+1} - E_{n+1.2n+2-j}) \in \alpha\,, \text{(positive roots)}\,,$$

$$-e_j\,, \to \mathfrak{G}^-_j = \sqrt{2}(E_{n+1,j} - E_{2n+2-j,n+1}) \in \beta\,, \text{(negative roots)}\,.$$

The levels of positive roots are

$$f_{e_j - e_k} = k - j\,,\; f_{e_j} = n - j + 1\,,\; f_{e_j+e_k} = 2n - k - j + 22n\,. \tag{3.13}$$

The Killing form normalized as in (A.24.I), (A.25.I) takes the form

$$(Z_1, Z_2) = \frac{1}{2}\mathrm{tr}\, Z_1 Z_2\,. \tag{3.14}$$

Then

$$(\mathfrak{G}^\pm_{jk}, \mathfrak{G}^\pm_{il}) = \delta_{ki}\delta_{jl}\,, \quad (\mathfrak{G}^+_j, \mathfrak{G}^-_k) = 2\delta_{jk}\,. \tag{3.15}$$

The GS-basis

The Weyl transformation Λ_1 in this basis is represented by the matrix

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -Id_{2n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}\,.$$

Taking into account its action on Z (3.11) we find that $\Pi_1 = \Pi_{B_{n-1}}$ and $\tilde{\Pi}^\vee = \Pi^\vee_{B_{n-1}}$. Then

$$\mathfrak{g}_{B_n} = \mathfrak{g}_0 + \mathfrak{g}_1\,, \quad \mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 + V\,, \;\; \tilde{\mathfrak{g}}_0 = \mathfrak{g}_{B_{n-1}}\,,$$

$$\dim\, \mathfrak{g}_{B_{n-1}} = (n-1)(2n-1)\,, \quad \dim\, V = 2(n-1) + 1\,, \quad \dim\, \mathfrak{g}_1 = 2(n-1) + 2\,,$$

where V is the vector representation of $\tilde{\mathfrak{g}}_0 = \mathbf{so}(2n-1)$. The invariant subalgebra $\mathfrak{g}_0$ is isomorphic to $\mathbf{so}(2n)$. Its Dynkin diagram is obtained from $\Pi^{ext}_{B_n}$ by dropping out the root α_n [5].

The space $V+\mathfrak{g}_1$ is represented by matrices of the form

$$
\begin{pmatrix}
a & \vec{x} & \alpha & \vec{y} & 0\\
(\vec{z})J & 0 & 0 & 0 & -(\vec{y})J\\
\beta & 0 & 0 & 0 & -\alpha\\
(\vec{w})J & 0 & 0 & 0 & -(\vec{x})J\\
0 & -\vec{w} & -\beta & -\vec{z} & -a
\end{pmatrix}.
$$

Since $\tilde{\mathfrak{g}}_0$ is generated by a trivial orbit ($l=1$), the GS-basis is formed by the Chevalley basis in $\mathfrak{g}_{B_{n-1}}$ (3.12) and the GS generators in $V+\mathfrak{g}_1$.

$$
V=\left\{\begin{array}{ll}
\mathfrak{t}^0_{1,k}=\frac{1}{\sqrt{2}}(E_{1,k}+E_{2n+1,k}-\dots) & \mathfrak{t}^0_{1,n+1+k}=\frac{1}{\sqrt{2}}(E_{1,n+1+k}+E_{2n+1,n+1+k}-\dots)\,,\\
\mathfrak{t}^0_{j+n+1,1}=\frac{1}{\sqrt{2}}(E_{j+n+1,1}+E_{j+n+1,2n+1}-\dots)\,, & \mathfrak{t}^0_{j,1}=\frac{1}{\sqrt{2}}(E_{j,1}+E_{j,2n+1}-\dots)\,,\\
\mathfrak{t}^0_{n+1,1}=(E_{n+1,1}+E_{n+1,2n+1}-\dots)\,, & \mathfrak{t}^0_{1,n+1}=(E_{1,n+1}+E_{2n+1,n+1}-\dots)\,,
\end{array}\right.
$$

$$
\mathfrak{g}_1=\left\{\begin{array}{ll}
\mathfrak{t}^1_{1,k}=\frac{1}{\sqrt{2}}(E_{1,k}-E_{2n+1,k}-\dots) & \mathfrak{t}^1_{1,n+1+k}=\frac{1}{\sqrt{2}}(E_{1,n+1+k}-E_{2n+1,n+1+k}-\dots)\,,\\
\mathfrak{t}^1_{j+n+1,1}=\frac{1}{\sqrt{2}}(E_{j+n+1,1}-E_{j+n+1,2n+1}-\dots)\,, & \mathfrak{t}^1_{j,1}=\frac{1}{\sqrt{2}}(E_{j,1}-E_{j,2n+1}-\dots)\,,\\
\mathfrak{t}^1_{n+1,1}=\frac{1}{\sqrt{2}}(E_{n+1,1}-E_{n+1,2n+1}-\dots)\,, & \mathfrak{t}^1_{1,n+1}=\frac{1}{\sqrt{2}}(E_{1,n+1}-E_{2n+1,n+1}-\dots)\,,\\
\mathfrak{h}^1_1=\sqrt{2}\mathrm{diag}(e_1,0\dots,0-e_1) & .
\end{array}\right.
$$

From (5.9.I) (5.35.I) and (3.14) we find the dual basis

$$
\mathfrak{T}^a_{jk}=\mathfrak{t}^a_{kj}\,,\quad j \text{ or } k\neq n+1\,,\quad \mathfrak{H}^1_1=\mathfrak{h}^1_1\,,\quad \mathfrak{T}^a_{n+1,1}=\frac{1}{2}\mathfrak{t}^a_{1,n+1}\,,\quad \mathfrak{T}^a_{1,n+1}=\frac{1}{2}\mathfrak{t}^a_{n+1,1}\,. \tag{3.16}
$$

and

$$
(\mathfrak{t}^a_{1j},\mathfrak{t}^b_{i1})=\left\{\begin{array}{ll}\delta^{(ab)}\delta_{(ji)} & i,j\neq n+1\,,\\ 2\delta^{(ab)}\delta_{(ji)} & i=n+1\,,\end{array}\right.\qquad (\mathfrak{h}^1_1,\mathfrak{h}^1_1)=4\,. \tag{3.17}
$$

The Lax operators and the Hamiltonians

Trivial bundles.

For trivial bundles the moduli space is described by the vector $\mathbf{u}=(u_1,\dots,u_n)$. For a trivial $\bar{G}=Spin(2n+1)$-bundles it means that

$$
\mathbf{u}\in\mathfrak{H}_{B_n}/W_{B_n}\ltimes(\tau Q^\vee\oplus Q^\vee)\,, \tag{3.18}
$$

where W_{B_n} is defined in (3.4). For trivial $SO(2n+1)$-bundles we have

$$
\mathbf{u}\in\mathfrak{H}_{B_n}/W_{B_n}\ltimes(\tau P^\vee\oplus P^\vee)\,. \tag{3.19}
$$

The dual variables $\mathbf{v}=(v_1,\dots,v_n)$ $v_j\in\mathbb{C}$ are the same in the both cases. The standard CM Lax operator in the Chevalley basis is

$$
L^{CM}_{B_n}(z)=\sum_{j=1}^n(v_j+S_{0,j}E_1(z))\mathfrak{E}_j+\sum_{j\neq k}S_{jk}\phi(u_j-u_k,z)\mathfrak{G}^-_{jk} \tag{3.20}
$$

$$
+\sum_{j\neq k}S_{j,-k}\phi(u_j+u_k,z)\mathfrak{G}^+_{j,k}+\sum_j S^\pm_j\phi(u_j,z)\mathfrak{G}^\pm_j\,,
$$

where $\mathfrak{E}_j = \mathrm{diag}(0,\dots,1,0,\dots,0,-1,0,\dots,0)$. The quadratic Hamiltonian after the reduction with respect to the Cartan subgroup takes the form (see (3.15))

$$H^{CM}_{B_n} = \frac{1}{2}\sum_{j=1}^{n} v_j^2 - \frac{1}{2}\sum_{j\neq k}((S_{jk}S_{kj}E_2(u_j-u_k) + S_{j,-k}S_{k,-j}E_2(u_j+u_k)) - \sum_j S_j^+ S_j^- E_2(u_j,z)\,. \tag{3.21}$$

Thus, we have two types of the standard CM systems with the same Hamiltonians and different configuration spaces described by (3.18) and (3.19).

Nontrivial bundles

The GS-basis is described above. Now $\tilde{\mathbf{u}} = (u_2,u_3,\dots,u_n)$ belongs to $\mathfrak{H}_{B_{n-1}}$. In fact, $\tilde{\mathbf{u}}$ belongs to one of fundamental domains in $\mathfrak{H}_{B_{n-1}}$ (3.18), (3.19) under the action of the coweight and the coroot lattices. Taking into account (3.13) and $h=2n$ we find from the general prescription

$$L_1(z) = S_1^1\phi(\frac{1}{2},z)\mathfrak{h}_1^1 + S^1_{n+1,1}\mathbf{e}\,(\frac{z}{2})\phi(\frac{1+\tau}{2},z)\mathfrak{t}^1_{1,n+1} + S^1_{1,n+1}\mathbf{e}\,(-\frac{z}{2})\phi(\frac{1+\tau}{2},z)\mathfrak{t}^1_{n+1,1} +$$

$$\sum_{k=2}^{n}\Big(S^1_{k,1}\mathbf{e}\,((\frac{k-1}{2n})z)\phi(\frac{(k-1)\tau}{2n}-u_k+\frac{1}{2},z)\mathfrak{t}^1_{1,k} + S^1_{1,k}\mathbf{e}\,((\frac{1-k}{2n})z)\phi(\frac{(1-k)\tau}{2n}+u_k+\frac{1}{2},z)\mathfrak{t}^1_{k,1}$$

$$+S^1_{n+1+k,1}\mathbf{e}\,(\frac{2n+1-k}{2n}z)\phi(\frac{(2n+1-k)\tau}{2n}-u_k+\frac{1}{2},z)\mathfrak{t}^1_{1,n+1+k}$$

$$+S^1_{1,n+1+k}\mathbf{e}\,(-\frac{2n+1-k}{2n}z)\phi(u_k-\frac{(2n+1-k)\tau}{2n}+\frac{1}{2},z)\mathfrak{t}^1_{k,1}\Big)\,,$$

$$L_0'(z) = S'_{n+1,1}\mathbf{e}\,(\frac{z}{2})\phi(\frac{\tau}{2},z)\mathfrak{t}^0_{1,n+1} + S'_{1,n+1}\mathbf{e}\,(-\frac{z}{2})\phi(-\frac{\tau}{2},z)\mathfrak{t}^0_{n+1,1} +$$

$$\sum_{k=2}^{n}\Big(S'_{k,1}\mathbf{e}\,((\frac{k-1}{2n})z)\phi(\frac{(k-1)\tau}{2n}-u_k,z)\mathfrak{t}^0_{1,k} + S'_{1,k}\mathbf{e}\,((\frac{1-k}{2n})z)\phi(\frac{(1-k)\tau}{2n}+u_k,z)\mathfrak{t}^0_{k,1}$$

$$+S'_{n+1+k,1}\mathbf{e}\,(\frac{2n+1-k}{2n}z)\phi(\frac{(2n+1-k)\tau}{2n}-u_k,z)\mathfrak{t}^0_{1,n+1+k}$$

$$+S'_{1,n+1+k}\mathbf{e}\,(-\frac{2n+1-k}{2n}z)\phi(u_k-\frac{(2n+1-k)\tau}{2n},z)\mathfrak{t}^0_{k,1}\Big)\,.$$

The Lax operator $\tilde{L}_0(z)$ coincides with (3.20) after the corresponding replacement of indices. Taking into account the pairing (3.17) after the reduction we come to the Hamiltonian

$$H = H^{CM}_{B_{n-1}} + H' + H_1\,,$$

$$-H' = S'_{n+1,1}S'_{1,n+1}E_2(\frac{\tau}{2}) + \frac{1}{2}\sum_{k=2}^{n}\Big(S'_{k,1}S'_{1,k}E_2(u_k-\frac{(k-1)\tau}{2n}) + S'_{n+1+k,1}S'_{1,n+1+k}E_2(\frac{(2n+1-k)\tau}{2n}-u_k)\Big)\,,$$

$$-H_1 = S^1_{n+1,1}S^1_{1,n+1}E_2(\frac{1+\tau}{2}) + \frac{1}{2}(S_1^1)^2E_2(\frac{1}{2}) +$$

$$\frac{1}{2}\sum_{k=2}^{n}\Big(S^1_{k,1}S^1_{1,k}E_2(u_k-\frac{(k-1)\tau}{2n}-\frac{1}{2}) + S'_{n+1+k,1}S'_{1,n+1+k}E_2(\frac{(2n+1-k)\tau}{2n}-u_k-\frac{1}{2})\Big)\,.$$

Again, we have two types of systems with the same Hamiltonians and different configuration spaces described by (3.18) and (3.19), where u_1 is omitted.

4 Sp(n) - C_n root system.

Roots, weights and bases.

The algebra Lie $\mathfrak{g}_{C_n}$ has rank n and $\dim(\mathfrak{g}_{C_n}) = 2n^2+n$. The system of simple roots Π_{C_n} is defined in (3.5). The minimal root is

$$-\alpha_0 = 2e_1 = 2\sum_{j=1}^{n-1}\alpha_j + \alpha_n\,. \tag{4.1}$$

There are $2n$ long roots $2e_{\pm j}$ and $2n(n-1)$ short roots $\pm e_j \pm e_k$, $j \neq k$. The Weyl group W_{C_n} of $R(C_n)$ coincides with W_{B_n} (3.4).

The levels of positive roots (A.3.I) are

$$f_{e_j-e_k} = k-j\,,\ f_{2e_j} = 2n-2j+1\,,\ f_{e_j+e_k} = 2n-k-j+1\,. \tag{4.2}$$

The simple coroots $\Pi^\vee_{C_n} = \Pi_{B_n}$ (3.1) generates the coroot lattice

$$Q^\vee_{C_n} = Q_{B_n} = \{\gamma = \sum_{j=1}^{n} m_j e_j \,|\, m_j \in \mathbb{Z}\}\,. \tag{4.3}$$

Since$\rho^\vee_{C_n} = (n-\frac{1}{2})e_1 + (n-\frac{3}{2})e_2 + \ldots + \frac{3}{2}e_{n-1} + \frac{1}{2}e_n$, and $h = 2n$

$$\kappa_{C_n} = \rho/h = (\frac{1}{2}-\frac{1}{4n})e_1 + (\frac{1}{2}-\frac{3}{4n})e_2 + \ldots + \frac{3}{4n}e_{n-1} + \frac{1}{4n}e_n\,. \tag{4.4}$$

The dual to Π_{C_n} the system of the fundamental coweights takes the form

$$\{\varpi^\vee_j = e_1 + e_2 + \ldots + e_j\,,\ j = 1,\ldots,n-1\,,\ \varpi^\vee_n = \frac{1}{2}\sum_{j=1}^{n} e_j\}\,.$$

It generates the coweight lattice

$$P^\vee_{C_n} = \{\gamma = \sum_{j=1}^{n}\mathbb{Z}e_j + \mathbb{Z}(\frac{1}{2}\sum_{j=1}^{n}e_j)\}\,. \tag{4.5}$$

The factor-group $P^\vee_{C_n}/Q^\vee_{C_n} \sim \mu_2$ is isomorphic to the center of $Sp(n)$. The center is generated by the coweight $\varpi^\vee_n$ ($\zeta = \mathbf{e}(\varpi^\vee_n)$). It follows from (A.17.I) and (4.1) that the fundamental alcove has the vertices

$$C_{alc} = (0, \frac{1}{2}\varpi^\vee_1, \ldots, \frac{1}{2}\varpi^\vee_{n-1}, \varpi^\vee_n)\,.$$

We use this expression to find λ_n corresponding to $\xi = \varpi^\vee_n$ (3.10.I). λ_n acts on the vertices of C_{alc} and, in this way, on $\varpi^\vee_j$, as

$$\lambda_n : (\varpi^\vee_1 \leftrightarrow \varpi^\vee_{n-1}\,, \varpi^\vee_2 \leftrightarrow \varpi^\vee_{n-2}\,, \ldots, \varpi^\vee_n \leftrightarrow 0)\,.$$

Then its action on $\Pi^{ext}_{C_n}$ assumes the form

$$\lambda_n : (\alpha_0 \leftrightarrow \alpha_n\,, \alpha_2 \leftrightarrow \alpha_{n-2}\,, \ldots \alpha_n \leftrightarrow\ \alpha_0,)\,.$$

The action of λ_n on the canonical basis $(e_1, e_2, \ldots, e_n)$ takes the form

$$\lambda_n : (e_1 \leftrightarrow -e_n\,, e_2 \leftrightarrow -e_{n-1}\,, \ldots)\,. \tag{4.6}$$

If $\mathbf{u} = \sum_j u_j e_j$, then $\lambda_n : (u_1 \leftrightarrow -u_n \,, u_2 \leftrightarrow -u_{n-1} \,, \ldots)$. We define the invariant vector $\tilde{\mathbf{u}}$ in the invariant basis $\tilde{e}_1 = e_1 - e_n \,,\ \tilde{e}_2 = e_2 - e_{n-1} \,, \ldots, \tilde{e}_l = e_l - e_{l+1}$, $l = \left[\frac{n}{2}\right]$

$$\tilde{\mathbf{u}} = \sum_{j=1}^{l} u_j \tilde{e}_j \,. \tag{4.7}$$

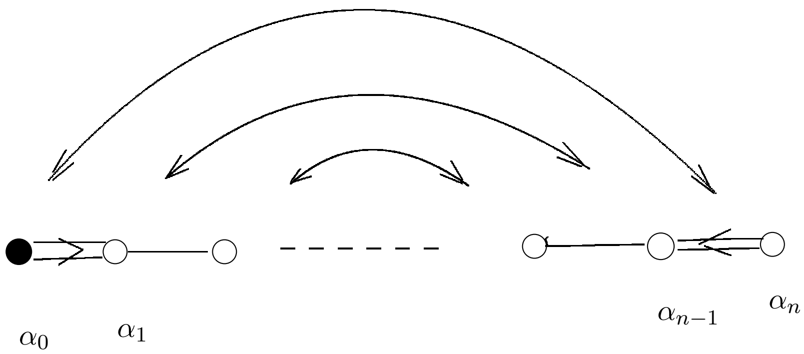


Fig.3 C_n and λ_n action

Bases.

The Chevalley basis

The Chevalley basis in $\mathfrak{g}_{C_n}$ is generated by the simple coroots $\Pi^{\vee}_{C_n} = \Pi_{B_n}$ (3.1) in $\mathfrak{H}$ and by the root subspaces. It is convenient to define the Chevalley basis using a fundamental representation π_1 corresponding to the weight $\varpi_1 = e_1$. It has dimension $2n$. We define it as the Lie algebra of matrices $\{Z\}$ satisfying the constraints

$$Zq + qZ^T = 0 \,, \quad Z \in \pi_1 \,,$$

where q in the bilinear form

$$q = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \,,$$

and J is an n-th order anti-diagonal matrix (3.10). Then Z takes the form

$$Z = \begin{pmatrix} A & B \\ C & -\tilde{A} \end{pmatrix} \,, \quad B = \tilde{B} \,,\ C = \tilde{C} \,, \quad \tilde{X} = JX^T J \,. \tag{4.8}$$

The basis in $\mathfrak{H}_{C_n}$ is generated by the canonical basis $(e_1, \ldots, e_n)$. Then the canonical basis in $\pi_1(\mathfrak{H})$ is

$$\mathrm{diag}(e_1, \ldots, e_n, -e_n, \ldots, -e_1) \,.$$

The Killing form is similar to the B_n case (3.14) $(Z_1,Z_2)=\frac{1}{2}\mathrm{tr}\,Z_1Z_2$.

The root subspaces in π_1 are

$$\begin{array}{ll}(e_j-e_k) & \to \mathfrak{G}^-_{jk}=(E_{j,k}\in A-E_{2n+1-k,2n+1-j}\in\tilde{A}),\,,\\ (e_j+e_k) & \to \mathfrak{G}^+_{jk}=(E_{j,k+n}\in B+E_{n+1-k,2n+1-j}\in C)\,,\end{array} \tag{4.9}$$

$$j<k\ \sim\ \text{(positive roots)}\,,\quad j>k\ \sim\ \text{(negative roots)}$$

$$2e_j\to\mathfrak{G}^+_j=E_{j,j+n}\in B\,,\ \text{(positive roots)}\,,\quad -2e_j\to\mathfrak{G}^-_j=E_{j+n,j}\in C\ \text{(negative roots)}\,.$$

The Killing form in this basis is

$$(\mathfrak{G}^\pm_{jk},\mathfrak{G}^\pm_{il})=\delta_{ki}\delta_{jl}\,,\quad (\mathfrak{G}^+_k,\mathfrak{G}^-_j)=\frac{1}{2}\delta_{jk}\,. \tag{4.10}$$

The GS-basis

The transformation λ_n (4.6) in π_1 takes the form

$$\Lambda_n=\begin{pmatrix}0 & i\,Id_n\\ i\,Id_n & 0\end{pmatrix},\quad \lambda_n(Z)=\begin{pmatrix}-\tilde{A} & C\\ B & A\end{pmatrix} \tag{4.11}$$

Since $\lambda_n^2=Id$

$$\mathfrak{g}_{C_n}=\mathfrak{g}_0+\mathfrak{g}_1\,, \tag{4.12}$$

where

$$\mathfrak{g}_0=\left\{\begin{pmatrix}X & Y\\ Y & X\end{pmatrix}\ \begin{array}{|c}\tilde{X}=-X\\ \tilde{Y}=Y\end{array}\right\},\quad \mathfrak{g}_1=\left\{\begin{pmatrix}X & Y\\ -Y & -X\end{pmatrix}\ \begin{array}{|c}\tilde{X}=X\\ \tilde{Y}=Y\end{array}\right\},$$

$$\mathfrak{g}_0=\tilde{\mathfrak{g}}_0+V\,,\ \tilde{\mathfrak{g}}_0=\left\{\begin{pmatrix}X & 0\\ 0 & X\end{pmatrix}\right\},\ V=\left\{\begin{pmatrix}0 & Y\\ Y & 0\end{pmatrix}\right\}, \tag{4.13}$$

$$\dim\,\mathfrak{g}_{C_n}=n(2n+1)\,,\quad \dim\,\tilde{\mathfrak{g}}_0=\frac{1}{2}n(n-1)\,,\quad \dim\,V=\frac{1}{2}n(n+1)\,,\quad \dim\,\mathfrak{g}_1=n(n+1)\,.$$

A type of $\tilde{\mathfrak{g}}_0$ depends on a parity of n. Note, that $\Pi_1=A_{n-1}$ (**??**) is generated by roots $(\alpha_1,\dots,\alpha_{n-1})$. Let $\lambda_n|_{\mathfrak{H}_{A_{n-1}}}=\tilde{\lambda}$. We prove general Lemma about automorphisms of $\mathfrak{g}_{A_{n-1}}$ that will be applied to C_n and D_n algebras. Let $(e_1,\dots,e_n)$ be a canonical basis in $\mathfrak{H}_{A_{n-1}}$ and $E_{j,k}$, $(1\le j<k\le n)$ is the root basis $\mathfrak{g}_{A_{n-1}}$ =sl$(n,\mathbb{C})$. It follows from (4.6) and (4.11) that the action of $\tilde{\lambda}$ on the Chevalley basis of $\mathfrak{H}_{A_{n-1}}$ takes the form

$$\tilde{\lambda}:\left\{\begin{array}{l}(e_1,e_2\dots,e_n)\ \to\ (-e_n,-e_{n-1},\dots,-e_1)\\ E_{jk}\ \to\ -E_{n-k+1,n-j+1}\,.\end{array}\right.\,. \tag{4.14}$$

Lemma 4.1 *Under the action (4.14)* $\mathfrak{g}_{A_{n-1}}$ *is decomposed as*

$$\mathfrak{g}_{A_{n-1}}=\tilde{\mathfrak{g}}_0+\tilde{\mathfrak{g}}_1\,,$$

where

•

$$\tilde{\mathfrak{g}}_0=\left\{\begin{array}{ll}\mathfrak{g}_{D_{\frac{n}{2}}} & n-\text{even}\,,\\ \mathfrak{g}_{B_{\frac{n-1}{2}}} & n-\text{odd}\,.\end{array}\right.$$

with the defined below Chevalley bases (4.17), (4.19).

• $\tilde{\mathfrak{g}}_1$ *is a space of traceless symmetric matrices of order* n $(\dim(\tilde{\mathfrak{g}}_1)=\frac{1}{2}n(n+1)$, *and* $\tilde{\mathfrak{g}}_0$ *acts on* $\tilde{\mathfrak{g}}_1$ *by commutators.*

Proof
Let $X \in$sl$(n,\mathbb{C})$ be a traceless matrix of order n and

$$\tilde{X} = JX^TJ\,, \quad J_{ik} = \delta_{i,n-1-k}\,.$$

It follows from (4.14) that $\tilde{\lambda}(X) = -\tilde{X}$ and the invariant subalgebra

$$\tilde{\mathfrak{g}}_0 = \mathfrak{g}_{A_{n-1}} + \tilde{\lambda}(\mathfrak{g}_{A_{n-1}})$$

is the algebra of anti-invariant matrices with respect to the symmetric form J_{jk}

$$XJ + JX^T = 0\,.$$

Thus, $\tilde{\mathfrak{g}}_0$ is the Lie algebra of orthogonal matrices SO$(n,\mathbb{C})$ and we come to the first statement.

Similarly, for $B \in \tilde{\mathfrak{g}}_1$

$$YJ - JY^T = 0\,. \tag{4.15}$$

It means that $\tilde{\mathfrak{g}}_1$ is the space of complex symmetric matrices with respect to the secondary diagonal. In these terms the commutation relations in A_{n-1} =sl$(n,\mathbb{C})$ assume the form

$$[\tilde{\mathfrak{g}}_0,\tilde{\mathfrak{g}}_0] \subset \tilde{\mathfrak{g}}_0\,, \quad [\tilde{\mathfrak{g}}_0,\tilde{\mathfrak{g}}_1] \subset \tilde{\mathfrak{g}}_1\,, \quad [\tilde{\mathfrak{g}}_1,\tilde{\mathfrak{g}}_1] \subset \tilde{\mathfrak{g}}_0\,.$$

We construct the Chevalley basis in the invariant subalgebras. The $\tilde{\lambda}$ on the simple roots of A_{n-1} is

$$\alpha_1 \leftrightarrow \alpha_{n-1}\,, \alpha_2 \leftrightarrow \alpha_{n-2}\,, \ldots, \begin{cases} \alpha_{\frac{n}{2}-1} \leftrightarrow \alpha_{\frac{n}{2}+1} & n\,\text{even}\,, \\ \alpha_{\frac{n}{2}} \leftrightarrow \alpha_{\frac{n}{2}+1} & n\,\text{even}\,, \\ \alpha_{\frac{n-1}{2}-1} \leftrightarrow \alpha_{\frac{n-1}{2}+2} & n\,\text{odd}\,, \\ \alpha_{\frac{n-1}{2}} \leftrightarrow \alpha_{\frac{n-1}{2}+1} & n\,\text{odd}\,, \end{cases}$$

For $n = 2l$ $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_{D_l} = \mathbf{so}(2l,\mathbb{C})$ the Chevalley basis is formed by the invariant coroots $\{\tilde{\alpha}_j^\vee$, $(j = 1,\ldots,l)\}$, constructed from A_{2l-1} roots $\alpha_1,\ldots,\alpha_{2l-1}$, and the root spaces basis

$$\tilde{\Pi}_{D_l}^\vee = \{\tilde{\alpha}_1^\vee = \alpha_1 + \alpha_{2l-1}\,, \ldots, \tilde{\alpha}_{l-1}^\vee = \alpha_{l-1} + \alpha_{l+1}\,, \; \tilde{\alpha}_l^\vee = \alpha_{l-1} + 2\alpha_l + \alpha_{l+1}\} = \tag{4.16}$$

$$\{\tilde{\alpha}_1^\vee = e_1 - e_2 + e_{n-1} - e_n\,, \ldots \tilde{\alpha}_{l-1}^\vee = e_{\frac{n}{2}-1} - e_{\frac{n}{2}} + e_{\frac{n}{2}+1} - e_{\frac{n}{2}+2}\,, \tilde{\alpha}_l^\vee = e_{\frac{n}{2}-1} + e_{\frac{n}{2}} - e_{\frac{n}{2}+1} - e_{\frac{n}{2}+2}\}\,.$$

The simple roots dual to coroots are $\tilde{\alpha}_j = \frac{1}{2}\tilde{\alpha}_j^\vee$. The matrix $a_{jk} = \langle\tilde{\alpha}_j, \tilde{\alpha}_k^\vee\rangle$ is the Cartan matrix D_l. The Chevalley generators corresponding to the simple roots are

$$\tilde{E}_{\tilde{\alpha}_j} = E_{j,j+1} - E_{n-j,n-j+1}\,, \; j < l\,, \quad \tilde{E}_{\tilde{\alpha}_l} = E_{l-1,l+1} - E_{l,l+2}\,. \tag{4.17}$$

For $n = 2l+1$ the Chevalley basis of the $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_{B_l} = \mathbf{so}(2l+1,\mathbb{C})$-algebra (see Section 9) takes the form

$$\tilde{\Pi}_{B_l}^\vee = \{\tilde{\alpha}_1^\vee = \alpha_1 + \alpha_{2l}\,, \ldots, \tilde{\alpha}_{l-1}^\vee = \alpha_{l-1} + \alpha_{l+2}\,, \tilde{\alpha}_l^\vee = 2(\alpha_l + \alpha_{l+1})\} = \tag{4.18}$$

$$\{\tilde{\alpha}_1^\vee = e_1 - e_2 + e_{n-1} - e_n\,, \ldots \tilde{\alpha}_{l-1}^\vee = e_{\frac{n-1}{2}-1} - e_{\frac{n-1}{2}} + e_{\frac{n-1}{2}+1} - e_{\frac{n-1}{2}+2}\,, \tilde{\alpha}_l^\vee = 2(e_{\frac{n-1}{2}-1} - e_{\frac{n-1}{2}+2})\}\,.$$

$$\tilde{E}_{\tilde{\alpha}_j} = E_{j,j+1} - E_{n-j,n-j+1}\,, \; j < l\,, \quad \tilde{E}_{\tilde{\alpha}_l} = E_{l,l+1} - E_{l+1,l+2}\,, \; j = l\,. \tag{4.19}$$

The dual root system

$$\tilde{\Pi}_{B_l} = \{\tilde{\alpha}_1 = \frac{1}{2}\tilde{\alpha}_1^\vee\,, \ldots, \tilde{\alpha}_{l-1} = \frac{1}{2}\tilde{\alpha}_{n-1}^\vee\,, \tilde{\alpha}_l = \frac{1}{4}\tilde{\alpha}_l^\vee\}\,.$$

leads to the B_l Cartan matrix $a_{jk} = \langle\tilde{\alpha}_j, \tilde{\alpha}_k^\vee\rangle$. □

Remark 4.1 *In this Lemma we have found the coroot basis in the invariant subalgebra $\tilde{\mathfrak{H}}_0$ for the special cases C_l, D_l . The expressions (4.16), (4.18) for $\tilde{\alpha}_l^\vee$ replace the general formula (5.26.I).*

Basis in V and $\mathfrak{g}_1$

We have constructed a basis in $\tilde{\mathfrak{g}}_0$. Consider other component in (4.12), (4.13). The basis in $V = \{B\}$, where B satisfies (4.15), has the form

$$\mathfrak{t}^0_{j,k+n} = \frac{1}{\sqrt{2}}(E_{j,k+n} + E_{n+j,k} + E_{n-k+1,2n-j+1} + E_{2n-k+1,n-j+1})\,,\ (j \neq k)\,, \tag{4.20}$$

$$\mathfrak{t}^0_{j,n+j} = \frac{1}{\sqrt{2}}(E_{j,n+j} + E_{n+j,j})\,.$$

It is easy to find that the invariant subalgebra $\mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 + V$ is isomorphic to $\mathbf{gl}(n)$. The form of $\mathfrak{g}_0$ is read off from from the extended Dynkin diagram for $\Pi^{ext}_{C_n}$ (Fig.3) by dropping out the roots α_0 and α_n (see [5]).

The GS basis in $\mathfrak{g}_1$ is represented as

$$\mathfrak{h}^1_j = \frac{1}{\sqrt{2}}(e_j + e_{n-j+1} - e_{2n-j+1} - e_{n+j})\,,\quad (j = 1, \dots \left[\frac{n}{2}\right])\,,$$

$$\mathfrak{t}^1_{j,n+k} = \frac{1}{\sqrt{2}}(E_{j,n+k} + E_{n-k+1,2n-j+1} - E_{n+j,k} - E_{2n-k+1,n-j+1})\,,\ (j \neq k)\,, \tag{4.21}$$

$$\mathfrak{t}^1_{j,k} = \frac{1}{\sqrt{2}}(E_{j,k} + E_{n-k+1,n-j+1} - E_{n+j,n+k} - E_{2n-k+1,2n-j+1})\,,\quad (j \neq k)\,,$$

$$\mathfrak{t}^1_{j,n+j} = \frac{1}{\sqrt{2}}(E_{j,n+j} - E_{n+j,j})\,.$$

In terms of the GS basis the Killing form is

$$(\mathfrak{t}^a_{j,n+j}, \mathfrak{t}^b_{k,n+k}) = \frac{(-1)^a}{2}\delta_{jk}\delta^{a,b}\,,\quad (\mathfrak{t}^a_{s,r+n}, \mathfrak{t}^b_{j,k+n}) = (-1)^a\delta_{sk}\delta_{rj}\delta^{a,b}\,, \tag{4.22}$$

$$(\mathfrak{t}^1_{s,r}, \mathfrak{t}^1_{j,k}) = \delta_{sk}\delta_{rj}\,,\quad (\mathfrak{h}^1_j, \mathfrak{h}^1_k) = \delta_{jk}\,.$$

The Lax operators and the Hamiltonians.

Trivial bundles.

For trivial bundles the moduli space is described by the vector $\mathbf{u} = (u_1, \dots, u_n)$. If E is a $\bar{G} = Sp(n)$-bundle

$$\mathbf{u} \in \mathfrak{H}_{C_n}/W_{C_n} \ltimes (\tau Q^\vee_{C_n} \oplus Q^\vee_{C_n})\,. \tag{4.23}$$

For trivial $Sp(n)/\mu_2$-bundles

$$\mathbf{u} \in \mathfrak{H}/W_{C_n} \ltimes (\tau P^\vee_{C_n} \oplus P^\vee_{C_n})\,. \tag{4.24}$$

The dual variables $\mathbf{v} = (v_1, \dots, v_n)$ $v_j \in \mathbb{C}$ are the same in the both cases. The standard CM Lax operator in the Chevalley basis is

$$L(z) = \sum_{j=1}^{n}(v_j + S_{0,j}E_1(z))e_j + \sum_{j\neq k} S_{j,k}\phi(u_j - u_k, z)\mathfrak{G}^-_{j,k} \tag{4.25}$$

$$+\sum_{j<k} S_{j,k+n}\phi(u_j+u_k,z)\mathfrak{G}^+_{j,k}+\sum_{j>k} S_{j+n,k}\phi(-u_j-u_k,z)\mathfrak{G}^+_{j,k}$$
$$\sum_j (S_{j,j+n}\phi(2u_j,z)\mathfrak{G}^-_j+S_{j+n,j}\phi(-2u_j,z)\mathfrak{G}^+_j)\,.$$

The quadratic Hamiltonian after the reduction with respect to the Cartan subgroup takes the form (see (4.10))

$$H^{CM}_{C_n}=\frac{1}{2}\sum_{j=1}^n v_j^2-\frac{1}{2}\sum_{j\neq k}((S_{j,k}S_{k,j}E_2(u_j-u_k)+S_{j,k+n}S_{j+n,k}E_2(u_j+u_k))-\frac{1}{4}\sum_j S_{j,j+n}S_{j+n,j}E_2(2u_j,z)\,.$$

Thus, we have two types of the standard CM systems with the same Hamiltonians and different configuration spaces described by (4.23) and (4.24).

Nontrivial bundles

Moduli space

Consider a bundle with a characteristic class defined by $\zeta=\mathbf{e}\,(\varpi^\vee_n)$. The moduli space is defined by vectors $\tilde{\mathbf{u}}\in\tilde{\mathfrak{H}}_0$ (4.7).

Let $n=2l+1$ be an odd number. As it was explained above, $\tilde{\mathfrak{H}}_0$ is a Cartan subalgebra of $\mathfrak{g}_{B_l}$. The coroot lattice in $\tilde{\mathfrak{H}}_0$ (4.3)

$$\tilde{Q}^\vee=\{\gamma=\sum_j^l m_j\tilde{e}_j\,,\ \ \sum_j^l m_j \text{ is even}\} \tag{4.26}$$

is invariant sublattice in $Q^\vee_{C_n}$ (4.3). The coweight lattice in $\tilde{\mathfrak{H}}_0$ (3.8)

$$\tilde{P}^\vee=\{\gamma=\sum_{j=1}^l m_j\tilde{e}_j\,|\,m_j\in\mathbb{Z}\}\,.$$

Let $\tilde{W}=W_{B_l}$ be the Weyl group. A closer of the positive Weyl chamber for $\tilde{\mathbf{u}}=(u_1,\ldots,u_l)$ is

$$u_1\geq u_2\geq\ldots\geq u_l\geq 0\,. \tag{4.27}$$

There are two types of the moduli spaces defined as the quotient of $\tilde{\mathfrak{H}}_0$

$$\tilde{\mathfrak{H}}_0/(\tilde{W}\ltimes(\tau\tilde{Q}^\vee+\tilde{Q}^\vee)\,,\quad \tilde{\mathfrak{H}}_0/(\tilde{W}\ltimes(\tau\tilde{P}^\vee+\tilde{P}^\vee)\,.$$

For $n=2l$ the invariant subalgebra is $\mathfrak{g}_{D_l}$. The structure of lattices for this algebra will be described in Section 5. The invariant coroot sublattice is the same as above (4.26). The form of the coweight sublattice depends on a parity of l.

Let l be odd. In this case

$$\tilde{P}^\vee=\tilde{Q}^\vee+\mathbb{Z}(\frac{1}{2}\sum_j^l\tilde{e}_j)\,.$$

There is an intermediate sublattice

$$\tilde{P}^\vee_2=\{\gamma=\sum_{j=1}^l m_j\tilde{e}_j\,|\,m_j\in\mathbb{Z}\}\,. \tag{4.28}$$

Therefore, there are three types of the moduli spaces:

$$\tilde{\mathfrak{H}}_0/(\tilde{W} \ltimes (\tau\tilde{Q}^\vee + \tilde{Q}^\vee)\,, \quad \tilde{\mathfrak{H}}_0/(\tilde{W} \ltimes (\tau\tilde{P}^\vee + \tilde{P}^\vee)\,, \quad \tilde{\mathfrak{H}}_0/(\tilde{W} \ltimes (\tau\tilde{P}_2^\vee + \tilde{P}_2^\vee)\,.$$

For l even there are two type of coweight lattices

$$\tilde{P}^{L\vee} = \tilde{Q}^\vee + \mathbb{Z}(\frac{1}{2}\sum_j^l \tilde{e}_j)\,, \quad \tilde{P}^{R\vee} = \tilde{Q}^\vee + \mathbb{Z}\frac{1}{2}(\sum_j^{l-1} \tilde{e}_j - \tilde{e}_l)$$

and the intermediate sublattice $\tilde{P}_2^\vee$ (4.28). In this case there are four types of the moduli spaces.

Lax operator

Using the general prescription for Lax operators (6.15.I), (6.17.I), (6.18.I), the GS basis (4.20), (4.21), and (4.2) we define L in our case. The invariant operator $\tilde{L}_0(z)$ is the Lax operator of the SO(n) CM system (3.20) and (6.26).

$$L_1(z) = \sum_{j=1}^n \Big(S_j\phi(\frac{1}{2}, z)\mathfrak{h}_j^1 + \frac{1}{2}S_{j,j+n}\mathbf{e}\,(\frac{2n+1-2j}{2n}z)\phi(\frac{(2n+1-2j)\tau}{2n} + \frac{1}{2} - 2u_j, z)\mathfrak{t}^1_{j,j+n}\Big)$$

$$+\sum_{j\neq k}^n \Big(S_{j,n+k}\mathbf{e}\,(\frac{2n+1-j-k}{2n}z)\phi(\frac{(2n+1-j-k)\tau}{2n} - u_j - u_k + \frac{1}{2}, z)\mathfrak{t}^1_{j,n+k} \tag{4.29}$$

$$+S_{j,k}\mathbf{e}\,(\frac{k-j}{n}z)\phi(\frac{(k-j)\tau}{n} - u_j + u_k + \frac{1}{2}, z)\mathfrak{t}^1_{k,j}\Big)\,,$$

$$L_0'(z) = \sum_{j\neq k}^n S'_{j,k+n}\mathbf{e}\,(\frac{2n+1-j-k}{2n}z)\phi(\frac{(2n+1-j-k)\tau}{2n} - u_j - u_k, z)\mathfrak{t}^0_{j,k+n}$$

$$+\frac{1}{2}\sum_{j=1}^n S'_{j,n+j}\mathbf{e}\,(\frac{2n+1-2j}{2n}z)\phi(\frac{(2n+1-2j)\tau}{2n} - 2u_j, z)\mathfrak{t}^0_{j,n+j}\,. \tag{4.30}$$

In these expressions the identification $u_j = -u_{n+1-j}$ as a result of λ_n action is assumed.

Hamiltonian

Due to the gradation the Hamiltonian contain three terms

$$H = \tilde{H}_0 + H_0' + H_1\,,$$

where $\tilde{H}_0$ is the CM Hamiltonian related to SO($n,\mathbb{C}$) (3.21), (6.27). To define H_0' and H_1 one should take into account the Killing form (4.22). Then from (4.29) and (4.30) we find

$$H_1 = \frac{1}{2}\sum_{j=1}^n \Big(-S_j^2 E_2(\frac{1}{2}) + S^2_{j,j+n}E_2(\frac{(2n+1-2j)\tau}{2n} - 2u_j + \frac{1}{2})\Big)$$

$$-\frac{1}{2}\sum_{j<k}^n \Big(-S_{j,k+n}S_{k,j+n}E_2(\frac{(2n+1-j-k)\tau}{2n} - u_j - u_k - \frac{1}{2}) + S_{j,k}S_{k,j}E_2(\frac{(k-j)\tau}{n} - u_j + u_k - \frac{1}{2})\Big)\,,$$

$$H_0' = -\frac{1}{2}\sum_{j<k}^n \Big(S'_{j,k+n}S'_{k,j+n}E_2(\frac{(2n+1-j-k)\tau}{2n} - u_j - u_k) + \frac{1}{2}(S'_{j,j+n})^2 E_2(\frac{(2n+1-2j)\tau}{2n} - 2u_j)\Big)\,.$$

As above $u_j = -u_{n-j+1}$.

5 SO$(2n)$, Spin$(2n)$. General construction

Roots and weights.

The Lie algebra D_n has dimension $2n^2 - n$ and rank n. The universal covering group $\bar{G}$ is Spin$(2n)$ and G^{ad} =SO$(2n)/\mu_2$. In terms of the canonical basis on $\mathfrak{H}_{D_n}$ $(e_1, e_2, \ldots, e_n)$ simple roots of D_n are $\Pi_{D_n} = \{\alpha_j = e_j - e_{j+1}\,,\ j = 1, \ldots, n-1\,,\ \alpha_n = e_1 + e_n\}$, and the minimal root is

$$\alpha_0 = -e_1 - e_2 = -(\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n)\,. \tag{5.1}$$

The roots of system R_{D_n} have the same length and $\sharp R = 2n(n-1)$. The positive roots are $R^+ = \{(e_j \pm e_k)\,,\ j,k = 1, \ldots, n\,,\ (j > k)\}$. The levels of positive roots are

$$f_{e_j - e_k} = k - j\,,\quad f_{e_j + e_k} = 2n - k - j\,. \tag{5.2}$$

The Weyl group W_{D_n} of R_{D_n} is the semidirect product of permutations S_n and the sign changes

$$S_n \ltimes \text{(the sign changes}\, e_j \to -e_j)\,,\ (\prod_{j=1}^{n}(\pm 1)_j = 1)\,. \tag{5.3}$$

The root and coroot systems coincide and R_{D_n} generates the root lattice

$$Q_{D_n} = \{\gamma = \sum_{j=1}^{n} m_j e_j \,|\, \sum_{j=1}^{n} m_j \text{ is even}\}\,. \tag{5.4}$$

The half-sum of positive D_n coroots is $\rho_{D_n} = (n-1)e_1 + (n-2)e_2 + \ldots + 2e_{n-2} + e_{n-1}$, and since the Coxeter number $h = 2n - 2$

$$\kappa = \rho/h = \frac{1}{2}e_1 + (\frac{1}{2} - \frac{1}{2(n-1)})e_2 + \ldots + (\frac{1}{2} - \frac{j-1}{2n-2})e_j + \ldots \frac{1}{2n-2}e_{n-1}\,. \tag{5.5}$$

The system of the fundamental weights takes the form

$$\varpi_j = e_1 + e_2 + \ldots + e_j\,,\ j = 1, \ldots, n-2\,,$$

$$\varpi_{n-1} = \frac{1}{2}(e_1 + \ldots + e_{n-1} - e_n)\,,\quad \varpi_n = \frac{1}{2}(e_1 + \ldots + e_{n-1} + e_n)\,.$$

The weights ϖ_1, ϖ_{n-1}, and ϖ_n are the highest weights of the vector $\underline{2n}$, right spinors $\underline{(2^{n-1})}^R$ and left spinors $\overline{(2^{n-1})}^L$ representations.

We find from (A.17.I) and (5.1) that the fundamental alcove has the vertices

$$C_{alc} = (0, \varpi_1, \frac{1}{2}\varpi_2, \ldots, \frac{1}{2}\varpi_{n-2}, \varpi_{n-1}, \varpi_n)\,. \tag{5.6}$$

The Chevalley basis

The Chevalley basis is convenient to define using a fundamental representation π_1 corresponding to the weight $\varpi_1 = e_1$. It has dimension $2n$. If the symmetric form is represented by the anti-diagonal matrix as for B_n then $Z \in$**so**$(2n)$ takes the form

$$Z = \begin{pmatrix} A & B \\ C & -\tilde{A} \end{pmatrix}\,,\quad B = -\tilde{B}\,,\ C = -\tilde{C}\,,\ \tilde{X} = JX^T J\,. \tag{5.7}$$

The basis in $\mathfrak{H}_{D_n}$ is generated by the simple roots Π_{D_n} in $\mathfrak{H}$ (or by the canonical basis $(e_1,\dots,e_n)$). In π_1 the canonical basis in $\mathfrak{H}_{D_n}$ is $\mathrm{diag}(e_1,\dots,e_n,-e_n,\dots,-e_1)$.

The root subspaces in π_1 are

$$\begin{array}{lll} (e_j-e_k)\,,\ j<k & \to \mathfrak{G}^-_{jk}=(E_{j,k}(\in A)-E_{2n+1-k,2n+1-j}(\in\tilde{A}),, & \text{(positive roots)} \\ (e_j-e_k)\,,\ j>k & \to \mathfrak{G}^-_{jk}=(E_{j,k}(\in A)-E_{2n+1-k,2n+1-j}(\in\tilde{A})),, & \text{(negative roots)}\,, \end{array} \tag{5.8}$$

$$\begin{array}{lll} (e_j+e_k)\,,\ j<k & \to \mathfrak{G}^+_{jk}=(E_{j,k+n}-E_{n+1-k,2n+1-j})\in B & \text{(positive roots)}\,, \\ (e_j+e_k)\,,\ j>k & \to \mathfrak{G}^+_{jk}=(E_{j+n,k}-E_{2n+1-k,n+1-j})\in C & \text{(negative roots)}\,. \end{array}$$

The Killing form (A.25.I) takes the form

$$(\mathfrak{G}^{\pm}_{jk},\mathfrak{G}^{\pm}_{il})=\delta_{jl}\delta_{ik}\,. \tag{5.9}$$

6 SO$(2n)$, Spin$(2n)$, $n=2l+1$.

Lattices and characteristic classes.

It follows from (5.4) that

$$\varpi_j\in Q(D_n)\ \text{for}\ 1<j<n-1\,,\quad \varpi_1\sim 2\varpi_n\,,\ \varpi_{n-1}\sim 3\varpi_n\,,\ mod\,Q(D_n)\,.$$

It means that

$$P(D_n)=Q(D_n)+\mathbb{Z}\varpi_n\sim Q(D_n)+\mathbb{Z}\varpi_{n-1}\,. \tag{6.1}$$

The weight lattice $P(D_n)$ contains a sublattice $P_2(D_n)$ of index two

$$P_2(D_n)=Q(D_n)+\mathbb{Z}\varpi_1=\{\sum_{j=1}^n m_je_j\,|\,m_j\in\mathbb{Z}\}\,. \tag{6.2}$$

This lattice is self-dual $P_2(D_n)={}^L P_2(D_n)$ and isomorphic to the group of cocharcters $P_2(D_n)=t(SO(2n))$, (A.43.I), where $SO(2n)=Spin(2n)/\mu_2$. On the other hand, the weight lattice $P(D_n)$ is dual to the root lattice ${}^LP(D_n)=Q(D_n)$.

The center of $Spin(2n)$ for odd n is $\mathcal{Z}(\bar{G})=P_{D_n}/Q_{D_n}\sim\mu_4$. The group element $\zeta=\mathbf{e}\,(\xi)$ for $\xi=\varpi_n$ generates μ_4. Putting $\xi=\varpi_n$ and C_{alc} (5.6) in (3.10.I) we find λ_n

$$\lambda_n\ :\ \left\{\begin{array}{l}\varpi_n\to 0\to\varpi_{n-1}\to\varpi_1\to\varpi_n\,,\\ \varpi_j\to\varpi_{n-j}\,,\ 1<j<n-1\,.\end{array}\right\}\,,\quad \left\{\begin{array}{l}\alpha_0\to\alpha_n\to\alpha_1\to\alpha_{n-1}\to\alpha_0\,,\\ \alpha_j\to\alpha_{n-j}\,,\quad 1<j<n-1\,.\end{array}\right\}\,.$$

In π_1 the transformation takes the form

$$\lambda_n=\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & Id_{n-1}\\ Id_{n-1} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}$$

It acts on the basis elements in the space of $\underline{2n}$ representation as

$$\begin{pmatrix} 1\,, & 2\,, & \dots, & n-1\,, & n\,, & n+1\,, & n+2\,, & \dots, & 2n-1\,, & 2n\\ n\,, & n+2\,, & \dots, & 2n-1 & 2n\,, & 1\,, & 2\,, & \dots, & n-1\,, & n+1\end{pmatrix}\,. \tag{6.3}$$

In the canonical basis $(e_1, e_2, \dots, e_n)$ λ_n is represented by the matrix

$$\lambda_n \to \Lambda_{jk} = \left\{ \begin{array}{l} \delta_{j,n-k+1}\,, \ j < n \\ -\delta_{j,n-k+1}\,, \ j = n\,. \end{array} \right. \tag{6.4}$$

This transformation is an element of the Weyl group $W(SO(2n))$.

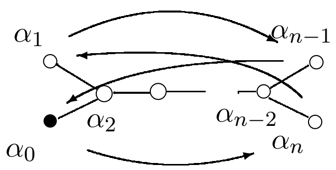


Fig.4 D_n, n-odd, λ_n

In a similar way the fundamental weight ϖ_{n-1} generates the Weyl transformation

$$\lambda_{n-1} : \left\{ \begin{array}{l} \varpi_n \to \varpi_1 \to \varpi_{n-1} \to 0 \to \varpi_n\,, \\ \varpi_j \to \varpi_{n-j}\,, \ 1 < j < n-1\,. \end{array} \right.$$

Thus, $\lambda_{n-1} = \lambda_n^{-1}$ and we will not consider this case.

Consider a subgroup of order two of $\mathcal{Z}(\bar{G})$ generated by $\xi = \varpi_1$. Acting as above we find

$$\lambda_1 : \left\{ \begin{array}{l} \varpi_1 \leftrightarrow 0\,, \\ \varpi_{n-1} \leftrightarrow \varpi_n\,, \\ \varpi_j \leftrightarrow \varpi_j\,, \ 1 < j < n-1\,. \end{array} \right.$$

In terms of roots the action assumes the form

$$\lambda_1 : \left\{ \begin{array}{l} \varpi_1 \leftrightarrow 0\,, \\ \varpi_{n-1} \leftrightarrow \varpi_n\,, \\ \varpi_j \leftrightarrow \varpi_j\,, \ 1 < j < n-1\,. \end{array} \right\}, \quad \left\{ \begin{array}{l} \alpha_0 \leftrightarrow \alpha_1\,, \\ \alpha_n \leftrightarrow \alpha_{n-1}\,, \\ \alpha_j \leftrightarrow \alpha_j\,, \ \ 1 < j \le n\,. \end{array} \right\}. \tag{6.5}$$

Explicitly, in π_1

$$\lambda_1 = \begin{pmatrix} 0 & & & & & 1 \\ & Id_{n-2} & & & 0 & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & 0 & & & Id_{n-2} & \\ 1 & & & & & 0 \end{pmatrix},$$

or

$$\left(\begin{array}{cccccccccc} 1\,, & 2\,, & \dots, & n-1\,, & n\,, & n+1\,, & n+2\,, & \dots, & 2n-1\,, & 2n \\ 2n\,, & 2\,, & \dots, & n-1\,, & n+1\,, & n\,, & n+2\,, & \dots, & 2n-1\,, & 1 \end{array} \right). \tag{6.6}$$

In the canonical basis $(e_1, e_2, \dots, e_n)$ λ_n is represented by the matrix

$$\lambda_1 \to \Lambda_{jk} = \left\{ \begin{array}{l} -\delta_{j,k}\,, \ j = 1, n \\ \delta_{j,k}\,, \ j \neq 1, n\,. \end{array} \right. \tag{6.7}$$

The GS-basis.

For n odd

$$\mathfrak{g}_{D_n} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3 \,, \quad (\lambda_n(\mathfrak{g}_k) = i^k \mathfrak{g}_k) \,, \tag{6.8}$$

where

$$\mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 + V \,.$$

In Π^{ext} there are one orbit of length 4 and $\frac{n-3}{2}$ orbits of length 2. The former orbit passes through α_0 and the latter orbits contain $\Pi_1 = A_{n-3}$. Since $n-3$ is even it follows from Lemma 10.1 that $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_{B_{\frac{n-3}{2}}}$. We will demonstrate it explicitly.

The subspaces in (6.8) can be read off from the λ_n-action. First we find that

$$\tilde{\mathfrak{g}}_0 = \left\{ Z = \begin{pmatrix} 0 & & & & 0 \\ & X & & 0 & \\ & & 0 & & \\ & 0 & & X & \\ 0 & & & & 0 \end{pmatrix} , \quad X = A^{(n-2)} - \tilde{A}^{(n-2)} \right\} ,$$

where $A^{(n-2)}$ is a matrix of order $n-2$, $\tilde{A}^{(n-2)} = J_{n-2} A^{(n-2)} J_{n-2}$.

$$\dim(\tilde{\mathfrak{g}}_0) = \frac{(n-2)(n-3)}{2} \,.$$

It is easy to see that $\tilde{\mathfrak{g}}_0$ has a type $B_{\frac{n-3}{2}}$. Namely, $\{X = A^{(n-2)} - \tilde{A}^{(n-2)}\} = \mathbf{so}(n-2)$. The invariant Cartan subalgebra $\tilde{\mathfrak{H}} \subset \tilde{\mathfrak{g}}_0$ has a basis

$$\tilde{e}_2 = e_2 - e_{n-1} \,, \; \tilde{e}_3 = e_3 - e_{n-2} \,, \ldots , \tilde{e}_l = e_l - e_{l+2} + e_{n+l} \,, \quad (l = \frac{n-3}{2}) \,. \tag{6.9}$$

An arbitrary element from $\tilde{\mathfrak{H}}_0$ has the form

$$\tilde{\mathbf{u}} = \mathrm{diag}\,(0, u_2, \ldots , u_l, 0, -u_l, \ldots , -u_2, 0, 0, u_2, \ldots , u_l, 0, -u_l, \ldots , -u_2, 0) \,. \tag{6.10}$$

The space V is generated by its GS basis. In following formulas $j = 2, \ldots , n-1$

$$\mathfrak{t}^0_{1j} = \frac{1}{2}(E_{1,j} + E_{n,n+j} + E_{2n,j} + E_{n+1,n+j}) - \ldots \,, \quad (j = 2, \ldots , n-1) \tag{6.11}$$

$$\mathfrak{t}^0_{1,n} = \frac{1}{2}(E_{1,n} + E_{n,2n} + E_{2n,n+1} + E_{n+1,1}) - \ldots \,,$$

$$\mathfrak{t}^0_{j,1} = \frac{1}{2}(E_{j,1} + E_{n+j,n} + E_{j,2n} + E_{n+j,n+1}) - \ldots \,, \quad (j = 2, \ldots , n-1) \,,$$

$$\mathfrak{t}^0_{j,n+k} = \frac{1}{\sqrt{2}}(E_{j,n+k} + E_{n+j,k}) - E_{n-k+1,2n-j+1} - E_{2n-k+1,n-j+1} \,, \quad (j,k = 2, \ldots , n-1) \,,$$

where $\ldots$ means the antisymmetric part of Z (5.7). The latter generators form the adjoint representation of $\tilde{\mathfrak{g}}_0 = \mathbf{so}(n-2)$. We have

$$\dim(V) = \frac{1}{2}(n-2)(n-3) + 2 \cdot (n-2) + 1 \,.$$

Let $\underline{(n-2)}$, $\underline{1}$ be a vector and a scalar representations of $\mathbf{so(n-2)}$. Then $\mathfrak{g}_0$ is decomposed as

$$\mathfrak{g}_0 = (\tilde{\mathfrak{g}}_0 = \mathbf{so(n-2)}) + 2 \times \underline{(n-2)} + \underline{1} + \underline{\frac{1}{2}(n-2)(n-3)} \,, \quad \dim(\mathfrak{g}_0) = n^2 - 3n + 3 \,.$$

It is isomorphic to $\mathbf{so}(n-1)+\mathbf{so}(n-1)+\underline{1}$. This algebra is obtained from the extended Dynkin diagram by dropping two middle roots. This procedure generates the automorphism of order four [5].

The Killing form in V is defined by (5.8.I). We write down its nonzero components

$$(\mathfrak{t}^0_{j_1,n+k_1},\mathfrak{t}^0_{j_2,n+k_2})=\delta_{j_1,k_2}\delta_{k_1,j_2}\,,\quad (\mathfrak{t}^0_{1,j},\mathfrak{t}^0_{j,1})=1\,,\quad (\mathfrak{t}^0_{1,n},\mathfrak{t}^0_{1,n})=-1\,. \tag{6.12}$$

Formally, the last pairing vanishes. However, one can consider instead nontrivial pairing $(\mathfrak{t}^0_{1,n},\mathfrak{t}^0_{n,1})$. The basic element $\mathfrak{t}^0_{n,1}$ is defined by the equivalent orbit passing through $E_{n,1}$. The sign minus in the last pairing arises due to the antisymmetry of matrix Z.

Consider the GS basis in $\mathfrak{g}_1$

$$\mathfrak{h}^1_1=\frac{1}{\sqrt{2}}(e_1+ie_n)-\ldots\,, \tag{6.13}$$

$$\mathfrak{t}^1_{1,j}=\frac{1}{2}(E_{1,j}+iE_{n,n+j}-E_{2n,j}-iE_{n+1,n+j})\ldots\,,\quad (j=2,\ldots,n-1)\,,$$

$$\mathfrak{t}^1_{1,n}=\frac{1}{2}(E_{1,n}+iE_{n,2n}-E_{2n,n+11}-iE_{n+1,1})-\ldots\,,$$

$$\mathfrak{t}^1_{j,1}=\frac{1}{2}(E_{j,1}+iE_{n+j,n}-E_{j,2n}-iE_{n+j,n+1})-\ldots\,,\quad (j=2,\ldots,n-1)\,,$$

$$\dim\,\mathfrak{g}_1=2(n-2)+2=2n-2\,.$$

The GS basis in $\mathfrak{g}_2$ is

$$\mathfrak{h}^1_1=\frac{1}{\sqrt{2}}(e_1-e_n)-\ldots\,,$$

$$\mathfrak{t}^2_{1j}=\frac{1}{2}(E_{1,j}-E_{n,n+j}+E_{2n,j}-E_{n+1,n+j})\ldots\,. \tag{6.14}$$

$$\mathfrak{t}^2_{1n}=\frac{1}{2}(E_{1,n}-E_{n,2n}+E_{2n,n+1}-E_{n+1,1})-\ldots\,,$$

$$\mathfrak{t}^2_{j,1}=\frac{1}{2}(E_{j,1}-E_{n+j,n}+E_{j,2n}-E_{n+j,n+1})-\ldots\,,\quad (j=2,\ldots,n-1)\,,$$

$$\mathfrak{t}^2_{j,n+k}=\frac{1}{\sqrt{2}}(E_{j,n+k}-E_{n+j,k})-\ldots\,,\quad (j,k=2,\ldots,n-1)\,,$$

$$\mathfrak{t}^2_{j,k}=\frac{1}{\sqrt{2}}(E_{j,k}+E_{n+1-k,n+1-j})-\ldots\,,\quad (j,k=2,\ldots,n-1)\,.$$

$$\dim\,\mathfrak{g}_2=2(n-2)+2+\frac{(n-2)(n-3)}{2}+\frac{(n-2)(n-1)}{2}=n^2-2n+1\,.$$

Finely, the GS basis in $\mathfrak{g}_3$ is

$$\mathfrak{h}^3_1=\frac{1}{\sqrt{2}}(e_1-ie_n)-\ldots\,, \tag{6.15}$$

$$\mathfrak{t}^3_{1j}=\frac{1}{2}(E_{1,j}-iE_{n,n+j}-E_{2n,j}+iE_{n+1,n+j})\ldots\,,\quad (j=2,\ldots,n-1)\,,$$

$$\mathfrak{t}^3_{1n}=\frac{1}{2}(E_{1n}-iE_{n,2n}-E_{2n,n+1}+iE_{n+1,1})-\ldots\,,$$

$$\mathfrak{t}^3_{j,1}=\frac{1}{2}(E_{j,1}-iE_{n+j,n}-E_{j,2n}+iE_{n+j,n+1})-\ldots\,,\quad (j=2,\ldots,n-1)\,,$$

$$\dim\,\mathfrak{g}_3 = 2(n-2)+2 = 2n-2\,.$$

It follows from (5.8.I) that there is a nontrivial pairing $(\mathfrak{g}_2,\mathfrak{g}_2)$ and $(\mathfrak{g}_1,\mathfrak{g}_3)$. For the basis we have

$$(\mathfrak{h}_1^1,\mathfrak{h}_1^3)=1\,,\quad (\mathfrak{t}_{1,j}^1,\mathfrak{t}_{j,1}^3)=1\,,\quad (\mathfrak{t}_{1,j}^3,\mathfrak{t}_{1,j}^1)=1\,,\quad (\mathfrak{t}_{1,n}^1,\mathfrak{t}_{1,n}^3)=-1\,, \tag{6.16}$$

$$(\mathfrak{h}_1^2,\mathfrak{h}_1^2)=1\,,\quad (\mathfrak{t}_{1j}^2,\mathfrak{t}_{j,1}^2)=1\,,\quad (\mathfrak{t}_{1,n}^2,\mathfrak{t}_{1,n}^2)=-1\,,$$

$$(\mathfrak{t}_{j_1,k_1}^2,\mathfrak{t}_{j_2,k_2}^2)=\delta_{j_1,k_2}\delta_{j_2,k_1}\,,\quad (\mathfrak{t}_{j_1,n+k_1}^2,\mathfrak{t}_{j_2,n+k_2}^2)=-\delta_{j_1,k_2}\delta_{j_2,k_1}\,. \tag{6.17}$$

Consider the GS-basis corresponding to λ_1 (6.6). In this case

$$\mathfrak{g}_{D_n}=\mathfrak{g}_0+\mathfrak{g}_1\,,\quad (\lambda_1(\mathfrak{g}_k)=(-1)^k\mathfrak{g}_k)\,,$$

$$\mathfrak{g}_0=\tilde{\mathfrak{g}}_0+V\,,\quad \tilde{\mathfrak{g}}_0=\mathfrak{g}_{B_{n-2}}=\mathbf{so}(2n-3)\,.$$

Then $Z\in\tilde{\mathfrak{g}}_0$ if

$$Z=\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0\\
0 & A^{(n-2)} & a_{jn} & a_{jn} & B^{(n-2)} & 0\\
0 & a_{nj} & 0 & 0 & -a_{jn}^T & 0\\
0 & a_{nj} & 0 & 0 & -a_{jn}^T & 0\\
 & C^{(n-2)} & -a_{nj}^T & -a_{nj}^T & -\tilde{A}^{(n-2)} & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)\in\mathbf{so}(2n-3)\,.$$

Here $A^{(n-2)}$, $B^{(n-2)}$ and $C^{(n-2)}$ are matrices of order $n-2$, $\tilde{A}^{(n-2)}=J(A^{(n-2)})^TJ$ and $\tilde{B}^{(n-2)}=-B^{(n-2)}$, $\tilde{C}^{(n-2)}=-C^{(n-2)}$.

The Cartan subalgebra $\tilde{\mathfrak{H}}_0\sim\mathfrak{H}_{B_{n-2}}$ has the form

$$\tilde{\mathfrak{H}}_0=\{\tilde{\mathbf{u}}=\mathrm{diag}(0,u_2,\ldots,u_{n-1},0,0,-u_{n-1},\ldots,-u_2,0\}\,. \tag{6.18}$$

For $Z\in V$ we have

$$Z=\left(\begin{array}{cccccc}
0 & a_{1j} & a_{1n} & b_{11} & -a_{j1}^T & 0\\
a_{j1} & 0 & 0 & 0 & 0 & a_{j1}\\
-a_{1n} & 0 & 0 & 0 & 0 & -b_{11}\\
-b_{11} & 0 & 0 & 0 & 0 & -a_{1n}\\
-a_{1j}^T & 0 & 0 & 0 & 0 & -a_{1j}^T\\
0 & a_{1j} & b_{11} & a_{1n} & -a_{j1}^T & 0
\end{array}\right)\,,$$

$$V=\underline{2(n-2)+1}+\underline{1}\,.$$

Here $\underline{2(n-2)+1}$ is a vector representation of $B_{n-2}=\mathbf{so}(2n-3)$. The GS-basis in V takes the form

$$\mathfrak{t}_{1,n}^0=\frac{1}{\sqrt{2}}(E_{1,n}+E_{2n,n+1})-\ldots\,,\quad \mathfrak{t}_{1,j}^0=\frac{1}{\sqrt{2}}(E_{1,j}+E_{2n,j})-\ldots\,,\ (j=2,\ldots,n-1)\,, \tag{6.19}$$

$$\mathfrak{t}_{n,1}^0=\frac{1}{\sqrt{2}}(E_{n,1}+E_{n+1,2n})-\ldots\,,\quad \mathfrak{t}_{j,1}^0=\frac{1}{\sqrt{2}}(E_{j,1}+E_{j,2n})-\ldots\,,\ (j=2,\ldots,n-1)\,.$$

As above, ... means the antisymmetric partners of the generators. Thus

$$\dim\,\mathfrak{g}_0=\dim\,\mathfrak{g}_{B_{n-2}}+\dim\,V=2n^2-5n+4\,.$$

The invariant subalgebra $\mathfrak{g}_0$ is isomorphic to $\mathbf{so(2n-2)}+\underline{1}$. It is obtained from the extended Dynkin diagram for $\mathbf{so(2n)}$ by dropping two roots roots α_1 and α_0. This procedure provides the second order automorphism [5] that we consider.

Non-vanishing components of the Killing form on V are

$$(\mathfrak{t}^0_{1,n},\mathfrak{t}^0_{n,1})=-1\,,\quad(\mathfrak{t}^0_{1,j},\mathfrak{t}^0_{j,1})=1\,. \tag{6.20}$$

Consider the GS-basis in $\mathfrak{g}_1$

$$\mathfrak{h}^1_1=e_1\,,\quad\mathfrak{h}^1_n=e_n\,, \tag{6.21}$$

$$\mathfrak{t}^1_{1,j}=\frac{1}{\sqrt{2}}(E_{1,j}-E_{2n,j})\dots\,,\quad\mathfrak{t}^1_{1,n}=\frac{1}{\sqrt{2}}(E_{1,n}-E_{2n,n+1})-\dots\,,$$

$$\mathfrak{t}^1_{n,1}=\frac{1}{\sqrt{2}}(E_{n,1}-E_{n+1,2n})-\dots\,,\quad\mathfrak{t}^1_{j,1}=\frac{1}{\sqrt{2}}(E_{j,1}-E_{j,2n})-\dots\,,$$

The Killing form on this basis assumes the form

$$(\mathfrak{h}^1_1,\mathfrak{h}^1_1)=1\,,\quad(\mathfrak{h}^1_n,\mathfrak{h}^1_n)=1\,,\quad(\mathfrak{t}^1_{1,n},\mathfrak{t}^1_{n,1})=1\,,\quad(\mathfrak{t}^1_{1,j},\mathfrak{t}^1_{j,1})=1\,. \tag{6.22}$$

The Lax operators and the Hamiltonians.

Trivial bundles.

The moduli space for the trivial $\bar{G}=Spin(2n)$- bundles is the quotient $\mathfrak{H}_{D_n}/(W_{D_n}\ltimes(\tau Q(D_n)+Q(D_n)))$, where W_{D_n} is defined in (5.3). It implies that

$$u_j\sim u_j+m_j+n_j\tau\,,\quad n_j\,,m_j\in\mathbb{Z}\,,\ \sum_{j=1}^n m_j\ \text{ and }\ \sum_{j=1}^n n_j\ \text{ is even} \tag{6.23}$$

(see (5.4)).

For trivial $G^{ad}=SO(2n)/\mu_2$-bundles the moduli space is $\mathfrak{H}_{D_n}/(W\ltimes(\tau P(D_n)+P(D_n)))$, where $P(D_n)$ is the D_n-weight lattice (6.1). In this case instead of the shifts (6.23) the moduli space is defined up to the shifts

$$u_j\sim u_j+m_j+n_j\tau\,,\quad n_j\,,m_j\in\mathbb{Z}\text{ or }\in\frac{1}{2}+\mathbb{Z},. \tag{6.24}$$

The intermediate situation arises for trivial $SO(2n)$-bundles. In this case $\mathbf{u}$ is defined up to the shifts

$$u_j\sim u_j+\tau m_j+n_j\,,\quad m_j,n_j\in\mathbb{Z}\,. \tag{6.25}$$

For all $Spin(2n)$, $SO(2n)$ $SO(2n)/\mu_2$ trivial bundles we have the same Lax operator

$$L(z)=\sum_{j=1}^n(v_j+S_{0,j}E_1(z))e_j+\sum_{j\neq k}^n S_{k,j}\phi(u_j-u_k,z)\mathfrak{G}^-_{j,k}+\sum_{j\neq k}^n S_{k+n,j}\phi(u_j+u_k,z)\mathfrak{G}^+_{j,k+n}\,, \tag{6.26}$$

and the same CM Hamiltonian

$$H=\frac{1}{2}\sum_{j=1}^n v_j^2-\sum_{j\neq k}^n S_{j,k}S_{k,j}E_2(u_j-u_k)-\sum_{j\neq k}^n S_{j,k+n}S_{j+n,k}E_2(u_j+u_k)\,, \tag{6.27}$$

but different configuration spaces (6.23), (6.24), (6.25).

Nontrivial bundles.

The moduli space.
Consider a bundle with a nontrivial characteristic class generated by the weight ϖ_n. The moduli space is a quotient of $\tilde{\mathfrak{H}}_0 = \{\tilde{\mathbf{u}} = \sum_{j=1}^{l} u_j \tilde{e}_j\}$, $l = \frac{n-3}{2}$, (see (6.10)), where $\tilde{\mathfrak{H}}_0$ is a Cartan subalgebra of the invariant algebra $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_{B_l} = \mathbf{so}(n-3)$.

Following (3.6) and (3.8) we construct invariant coroot and coweight sublattices

$$\tilde{Q}^\vee(B_l) = \{\sum_{j=1}^{l} m_j \tilde{e}_j\,,\ m_j \in \mathbb{Z}\,,\ \sum_{j=1}^{l} m_j - \text{even}\}\,,\quad \tilde{P}^\vee(B_l) = \{\sum_{j=1}^{l} m_j \tilde{e}_j\,,\ m_j \in \mathbb{Z}\}\,.$$

where $\{\tilde{e}_j\}$ is the invariant basis (6.9). Therefore, there are two types of the moduli spaces. They are defined up to the shifts by a vector $\tilde{\mathbf{u}}$

$$\tilde{\mathbf{u}} \sim \tilde{\mathbf{u}} + \tau\gamma_1 + \gamma_2\,,\quad \gamma_j \in \tilde{Q}^\vee(B_l)\,,\ \text{or}\ \tilde{P}^\vee(B_l)\,.$$

The nontrivial $Spin_{2n}$, SO$(2n)$, SO$(2n)/\mu_2$ - bundles with the characteristic class ϖ_n have the same moduli space as trivial $Spin_{n-2}$, SO$(n-2)$-bundles. Two fundamental domains describe these moduli spaces.

Let us define the moduli space of bundles with nontrivial characteristic class generated by the weight ϖ_1. In this case $\Pi_1 = \Pi_{D_{n-1}}$ and $\tilde{\Pi} = \Pi_{B_{n-2}}$ ($\tilde{\mathfrak{g}}_0 = \mathbf{so}(2n-3)$). Then $\tilde{\mathbf{u}} \in \mathfrak{H}_{B_{n-2}}$ (6.18). Two lattices $\tilde{Q}^\vee(B_{n-2})$, $\tilde{P}^\vee(B_{n-2})$ in $\mathfrak{H}_{B_{n-2}}$ (3.6), (3.8) defines two fundamental domains

$$\tilde{\mathbf{u}} \sim \tilde{\mathbf{u}} + \tau\gamma_1 + \gamma_2\,,\quad \gamma_j \in \tilde{Q}^\vee(B_{n-2})\,,\ \text{or}\ \tilde{P}^\vee(B_{n-2})\,.$$

They are the moduli spaces of trivial $Spin_{2n-3}$, SO$(2n-3)$-bundles.

The Lax operators and Hamiltonians.
In the GS-basis corresponding to the characteristic class ϖ_n the Lax operator is decomposed as

$$L = \tilde{L}_0 + L_0' + L_1 + L_2 + L_3\,,$$

where $\tilde{L}_0$ is a Lax operator corresponding to the trivial SO$(n-2)$ bundle (3.20). The other components take the form

$$L_0' = S_{1,n}'\mathbf{e}\,(z/2)\phi(\frac{1}{2}, z)\mathfrak{t}^0_{1,n} +$$

$$\sum_{j=2}^{n-1}\Big(S_{j,1}'\mathbf{e}\,(\frac{j-1}{2n-2}z)\phi(\frac{j-1}{2n-2}\tau - u_j, z)\mathfrak{t}^0_{1,j} + S_{1,j}'\mathbf{e}\,(\frac{1-j}{2n-2}z)\phi(\frac{1-j}{2n-2}\tau + u_j, z)\mathfrak{t}^0_{j,1}$$

$$+\sum_{k=1}^{n-2} S_{k,n+j}'\mathbf{e}\,(\frac{2n-k-j}{2n-2}z)\phi(\frac{2n-k-j}{2n-2}\tau - u_j - u_k, z)\mathfrak{t}^0_{j,k+n}\Big)\,.$$

Here u_j are elements of the diagonal matrix (6.10) $u_i = -u_{n-i}$, $(i = 2, \ldots, l)$.

$$L_{k=1,3} = S_1^{4-k}\phi(\frac{k}{4}, z)\mathfrak{h}_1^k + S_{1,n}^{4-k}\mathbf{e}(zk/4)\phi(k/4, z)\mathfrak{t}^k_{1,n}$$

$$+\sum_{j=2}^{n-1}\Big(S_{j,1}^{4-k}\mathbf{e}\,(\frac{j-1}{2n-2}z)\phi(\frac{j-1}{2n-2}\tau - u_j + \frac{k}{4}, z)\mathfrak{t}^k_{1,j} + S_{1,j}^{4-k}\mathbf{e}\,(\frac{1-j}{2n-2}z)\phi(\frac{1-j}{2n-2}\tau + u_j - \frac{k}{4}, z)\mathfrak{t}^k_{j,1}\Big)\,,$$

$$L_2 = S_{1,n}^2\mathbf{e}(z/2)\phi(1/2, z)\mathfrak{t}^2_{1,n} + S_1^2\mathbf{e}(z/2)\phi(1/2, z)\mathfrak{h}_1^2 +$$

$$\sum_{j=2}^{n-1}\Big(S^2_{j,1}\mathbf{e}\,(\frac{j-1}{2n-2}z)\phi(\frac{(j-1)\tau}{2n-2}-u_j+\frac{1}{2},z)\mathfrak{t}^2_{1,j}+S^2_{1,j}\mathbf{e}\,(\frac{1-j}{2n-2}z)\phi(\frac{1-j}{2n-2}\tau+u_j-\frac{1}{2},z)\mathfrak{t}^2_{j,1}+$$

$$\sum_{m=2,m\neq j}^{n-1}S^2_{m,j+n}\mathbf{e}\,(\frac{2n-m-j}{2n-2}z)\phi(\frac{2n-m-j}{2n-2}\tau-u_j-u_m-\frac{1}{2},z)\mathfrak{t}^2_{j,m+n}$$

$$+\sum_{m=2,m\neq j}^{n-1}S^2_{m,j}\mathbf{e}\,(\frac{m-j}{2n-2}z)\phi(\frac{(m-j)}{2n-2}\tau-u_j+u_m-\frac{1}{2},z)\mathfrak{t}^2_{j,m}\Big)\,.$$

After the symplectic reduction with respect to the Cartan subgroup $\tilde{\mathcal{H}}$ we come to Hamiltonians of integrable systems

$$H=\tilde{H}_0+H'_0+H_2\,,$$

where $\tilde{H}_0$ is the Hamiltonian of B_l CM system (3.21),

$$H'_0=\frac{1}{2}(L'_0(z),L'_0(z))|_{\text{const. part}}\,,\quad H_2=(L_1(z),L_3(z))+\frac{1}{2}(L_2(z),L_2(z))|_{\text{const. part}}\,.$$

From (6.12) and (6.17) we find

$$-H'_0=\frac{1}{2}\Big(-(S'_{1,n})^2E_2(\frac{1}{2})+\sum_{j=2}^{n-1}S'_{1,j}S'_{j,1}E_2(\frac{j-1}{2n-2}\tau-u_j)+\sum_{k\neq j}^{n-1}S'_{j,n+k}S'_{k,n+j}E_2(\frac{2n-k-j}{2n-2}\tau-u_j-u_k)\Big)\,,$$

$$H_2=(S^1_1S^3_1-S^1_{1,n}S^3_{1,n})E_2(1/4)-\frac{1}{2}(S^2_{1,n})^2E_2(\frac{1}{2})+\frac{1}{2}\sum_{j\neq m}^{n-1}S^2_{j,m}S^2_{m,j}E_2(\frac{m-j}{2n-2}\tau-u_j+u_m-\frac{1}{2})$$

$$+\sum_{j=2}^{n-1}(S^1_{j,1}S^3_{j,1}+S^3_{j,1}S^1_{j,1})E_2(\frac{1-j}{2n-2}\tau+u_j-\frac{1}{4})+\frac{1}{2}\sum_{j\neq m}^{n-1}(S^2_{j,m+n}S^2_{m+n,j}E_2(\frac{2n-m-j}{2n-2}\tau-u_j-u_m-\frac{1}{2})\,.$$

Consider systems corresponding to the characteristic class ϖ_1. In this case using the GS-basis (6.19), (6.21) we define $L=\tilde{L}_0+L'_0+L_1$, where $\tilde{L}_0$ is the Lax operator of B_{n-2} (3.20),

$$L'_0(z)=S'_{n,1}\mathbf{e}\,(\frac{1}{2}z)\phi(\tau/2,z)\mathfrak{t}^0_{1,n}+S'_{1,n}\mathbf{e}\,(-\frac{1}{2}z)\phi(\tau/2,z)\mathfrak{t}^0_{n,1}+$$

$$\sum_{j=2}^{n-1}\Big(S'_{1,j}\mathbf{e}\,(\frac{j-1}{2n-2}z)\phi(\frac{(j-1)\tau}{2n-2}+u_j,z)\mathfrak{t}^0_{1j}+S'_{j,1}\mathbf{e}\,(\frac{1-j}{2n-2}z)\phi(\frac{(1-j)\tau}{2n-2}-u_j,z)\mathfrak{t}^0_{j,1}\Big)\,,$$

$$L_1(z)=(S^1_1\mathfrak{h}^1_1+S^1_n\mathfrak{h}^1_n)\phi(\frac{1}{2},z)+(S^1_{1,n}\mathfrak{t}^1_{n,1}+S^1_{n,1}\mathfrak{t}^1_{1,n})\mathbf{e}\,(\frac{1}{2}z)\phi(\frac{1}{2}(1+\tau),z)$$

$$+\sum_{j=2}^{n-1}\Big(S^k_{1,j}\mathbf{e}\,(\frac{j-1}{2n-2})\phi(\frac{(j-1)\tau}{2n-2}-u_j+\frac{1}{2},z)\mathfrak{t}^1_{1j}+S^1_{j,1}\mathbf{e}\,(\frac{1-j}{2n-2})\phi(\frac{(1-j)\tau}{2n-2}+u_j+\frac{1}{2},z)\mathfrak{t}^1_{j,1}\Big)\,.$$

Then we come to the Hamiltonians $\tilde{H}_0=H_{B_{\frac{n-3}{2}}}$ (3.21) and

$$H'_0=-\frac{1}{2}S'_{1,n}S'_{n,1}\wp(\tau/2)+\sum_{j=2}^{n-1}S'_{1,j}S'_{j,1}E_2(\frac{(j-1)\tau}{2n-2}-u_j)\,,$$

$$-2H_1=((S^1_1)^2+(S^1_n)^2)E_2(\frac{1}{2})+\sum_{j=2}^{n-1}S^1_{1,j}S^1_{j,1}E_2(\frac{(j-1)\tau}{2n-2}+\frac{1}{2}-u_j)+S^1_{1,n}S^1_{n,1}E_2(\frac{1}{2}(1+\tau))\,.$$

7 SO$(2n)$, Spin$(2n)$, $n = 2l$.

Lattices and characteristic classes

For $n = 2l$

$$P(D_n) = Q(D_n) + \mathbb{Z}\varpi_a + \mathbb{Z}\varpi_b\,, \quad a, b = (1, n-1, n)\,,\ a \neq b\,. \tag{7.1}$$

The weight lattice $P(D_n)$ contains apart from $Q(D_n)$ three sublattices of index 2 generated by these weights

$$P^V(D_n) = Q(D_n) + \mathbb{Z}\varpi_1\,, \tag{7.2}$$

$$P^R(D_n) = Q(D_n) + \mathbb{Z}\varpi_{n-1}\,, \quad P^L(D_n) = Q(D_n) + \mathbb{Z}\varpi_n\,.$$

The sublattice $P^V(D_n)$ is self-dual. If $n = 8l$ then ${}^L(P^R(D_n)) = P^L(D_n)$. For $n = 8l + 4$ ${}^L(P^R(D_n)) = P^R(D_n)$, and ${}^L(P^L(D_n)) = P^L(D_n)$. In other words, we have the following hierarchy of the lattices

$$\begin{array}{ccccc} & & P & & \\ & \swarrow & \downarrow & \searrow & \\ P^R & & P^V & & P^L \\ & \searrow & \downarrow & \swarrow & \\ & & Q & & \end{array}$$

The center of $Spin(2n)$ for even n is $\mathcal{Z}(\bar{G}) \sim \mu_2 \times \mu_2$. The group element $\zeta = \mathbf{e}\,(\xi)$ for $\xi = \varpi_n$ generates one of the subgroups μ_2. Putting $\xi = \varpi_n$ and C_{alc} (5.6) in (3.10.I) we find λ_n

$$\lambda_n\ :\ \left\{ \begin{array}{l} 0 \leftrightarrow \varpi_n\,, \quad \varpi_1 \leftrightarrow \varpi_{n-1}\,, \\ \varpi_j \leftrightarrow \varpi_{n-j}\,,\ 1 < j < n-1\,. \end{array} \right.$$

In terms of roots the action assumes the form $\alpha_0 \leftrightarrow \alpha_n\,,\ \alpha_1 \leftrightarrow \alpha_{n-1}\,, \ldots\ \alpha_j \leftrightarrow \alpha_{n-j}\,,$
$1 < j < n-1$. In the representation π_1 (5.7) the λ_n action takes the form

$$\lambda_n = \begin{pmatrix} 0 & Id_n \\ Id_n & 0 \end{pmatrix}.$$

Its action on the indices of the basis $(e_1, e_2, \ldots, e_{2n})$ is

$$\lambda_n\ :\ \begin{pmatrix} 1 & j & n & n+1 & n+j & 2n \\ n+1 & n+j & 2n & 1 & j & n \end{pmatrix}. \tag{7.3}$$

In the canonical basis $(e_1, e_2, \ldots, e_n)$ λ_n in $\mathfrak{H}_{D_{2l}}$ is represented by the matrix

$$\lambda_n \to E_{jk} = -\delta_{j,n-k+1}\,.$$

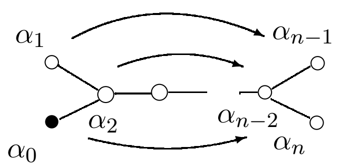


Fig.5 D_n, n-even, λ_n

Let us take $\xi = \varpi_{n-1}$. Then the corresponding Weyl transformation λ_{n-1} acts on C_{alc} as

$$\lambda_{n-1} \, : \left\{ \begin{array}{l} 0 \leftrightarrow \varpi_{n-1} \, , \quad \varpi_1 \leftrightarrow \varpi_n \, , \\ \varpi_j \leftrightarrow \varpi_{n-j} \, , \ 1 < j < n-1 \, , \end{array} \right.$$

or $\alpha_{n-1} \leftrightarrow \alpha_0$, $\alpha_n \leftrightarrow \alpha_1$, $\alpha_j \leftrightarrow \alpha_{n-j}$, $1 < j < n-1$.

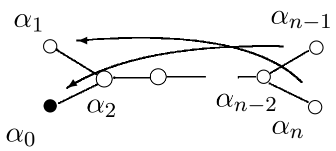


Fig.6 D_n, n-even, λ_{n-1}

In the representation π_1 (5.7) the λ_{n-1} action takes the form

$$\lambda_{n-1} = \left(\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Id_{n-2} & 0 \\ 1 & 0 & & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 1 \\ 0 & Id_{n-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right) .$$

It is conjugated to λ_n by the matrix

$$\left(\begin{array}{cccc} Id_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & Id_{n-1} \end{array} \right) .$$

We will not consider the corresponding GS basis.

The case $\xi = \varpi_1^\vee$ was already considered (6.6). In this case $\tilde{\mathfrak{g}}_0 = \mathbf{so}(2n-3)$.

The GS-basis.

The λ_n action on Z in π_1 takes the form

$$\lambda_n(Z) = \lambda_n \left(\begin{array}{cc} A & B \\ C & -\tilde{A} \end{array} \right) = \left(\begin{array}{cc} -\tilde{A} & C \\ B & A \end{array} \right) ,$$

Since $\lambda_n^2 = Id$

$$\mathfrak{g}_{D_n} = \mathfrak{g}_0 + \mathfrak{g}_1 \, , \tag{7.4}$$

where

$$\mathfrak{g}_0 = \left\{ \left(\begin{array}{cc} X & Y \\ Y & X \end{array} \right) \ \Big| \ \begin{array}{c} \tilde{X} = -X \\ \tilde{Y} = -Y \end{array} \right\} , \quad \mathfrak{g}_1 = \left\{ \left(\begin{array}{cc} X & Y \\ -Y & -X \end{array} \right) \ \Big| \ \begin{array}{c} \tilde{X} = X \\ \tilde{Y} = -Y \end{array} \right\} ,$$

where

$$\mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 + V \, , \quad \tilde{\mathfrak{g}}_0 = \left\{ \left(\begin{array}{cc} X & 0 \\ 0 & X \end{array} \right) \right\} , \quad V = \left\{ \left(\begin{array}{cc} 0 & Y \\ Y & 0 \end{array} \right) \right\} . \tag{7.5}$$

According to Lemma 9.1 $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_{D_{\frac{n}{2}}} = \mathbf{so}(n)$. The Cartan subalgebra $\tilde{\mathfrak{H}}_0$ in $\mathfrak{g}_0$ has the basis

$$\begin{array}{l} \tilde{e}_1 = \mathrm{diag}(1,0,\dots,0,-1,1,0,\dots,0,-1)\,, \\ \tilde{e}_2 = \mathrm{diag}(0,1,\dots,-1,0,0,1,\dots,-1,0)\,, \\ \dots\dots, \\ \tilde{e}_l = \mathrm{diag}(0,\dots,1,-1,\dots,0,0,\dots,1,-1,\dots,0)\,. \end{array} \tag{7.6}$$

It defines the moduli space of the $\mathrm{SO}(2n,\mathbb{C})$-bundles with characteristic class corresponding to ϖ_n

$$\tilde{\mathfrak{H}}_0 = \{\tilde{\mathbf{u}} = \sum_{j=1}^{l} u_j \tilde{e}_j\}\,. \tag{7.7}$$

From (7.3) the GS-basis in V takes the form

$$\mathfrak{t}^0_{j,n+k} = \frac{1}{\sqrt{2}}(E_{j,n+k} + E_{n+j,k} - E_{n-k+1,2n-j+1} - E_{2n-k+1,n-j+1})\,,\ \ (j,k=1,\dots,n\,,\ j\neq k) \tag{7.8}$$

with the Killing form

$$(\mathfrak{t}^0_{j_1,n+k_1},\mathfrak{t}^0_{j_2,n+k_2}) = \delta_{j_1,k_2}\delta_{k_1,j_2}\,. \tag{7.9}$$

The GS-basis in $\mathfrak{g}_1$ is

$$\mathfrak{h}^1_j = \frac{1}{\sqrt{2}}(e_j - e_{n+j} + e_{n-j+1} - e_{2n-j+1})\,,\ \ (j=1,\dots,l)\,,$$

$$\mathfrak{t}^1_{j,k} = \frac{1}{\sqrt{2}}(E_{j,k} - E_{n+j,n+k} + E_{n-k+1,n-j+1} - E_{2n-k+1,2n-j+1})\,,\ \ (j,k=1,\dots,l)\,, \tag{7.10}$$

$$\mathfrak{t}^1_{j,n+k} = \frac{1}{\sqrt{2}}(E_{j,n+k} - E_{n+j,k} - E_{n-k+1,2n-j+1} + E_{2n-k+1,n-j+1})\,,\ \ (j,k=1,\dots,l\,,\ j\neq k)\,,$$

with the Killing form

$$\begin{array}{c} (\mathfrak{h}^1_j,\mathfrak{h}^1_k) = \delta_{j.k}\,,\ \ (\mathfrak{t}^1_{j_1,k_1},\mathfrak{t}^1_{j_2,k_2}) = \delta_{j_1,k_2}\delta_{j_2,k_1}\,, \\ (\mathfrak{t}^1_{j_1,n+k_1},\mathfrak{t}^1_{j_2,n+k_2}) = -\delta_{j_1,k_2}\delta_{j_2,k_1}\,. \end{array} \tag{7.11}$$

The GS-basis and the Lax operator related to the characteristic class defined by ϖ_{n-1} are conjugated to the GS-basis and the Lax operator related to the characteristic class defined by ϖ_n. Therefore the corresponding Hamiltonians coincide.

Lax operators and Hamiltonians.

Consider a bundle with nontrivial characteristic class generated by the weight ϖ_n. The moduli space is a quotient of $\tilde{\mathfrak{H}}_0 = \{\tilde{\mathbf{u}} = \sum_{j=1}^{l} u_j\tilde{e}_j\}$, $\ l=\frac{n}{2}$, where $\tilde{\mathfrak{H}}_0$ is a Cartan subalgebra of the invariant algebra $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_{D_l}$ (7.7)

For l odd there are three types of invariant sublattices $\tilde{Q}(D_l)$ (5.4), $\tilde{P}(D_l)$ (6.1) and $\tilde{P}_2(D_l)$ (6.2). They define three types of moduli spaces for these bundles as the fundamental domains

$$\tilde{\mathbf{u}} \sim \tilde{\mathbf{u}} + \gamma_1\tau + \gamma_2\,,\ \ \gamma_j \in \tilde{Q}(D_l)\,,\ \text{or}\ \tilde{P}(D_l)\,,\ \text{or}\ \tilde{P}_2(D_l)\,. \tag{7.12}$$

For l even the invariant sublattices are $\tilde{Q}(D_l)$, $\tilde{P}^L(D_l)$, $\tilde{P}^R(D_l)$ and $\tilde{P}^V(D_l)$ (7.2). Therefore, in this case there are four types of the moduli spaces.

$$\tilde{\mathbf{u}} \sim \tilde{\mathbf{u}} + \gamma_1\tau + \gamma_2\,,\ \ \gamma_j \in \tilde{Q}(D_l)\,,\ \text{or}\ \tilde{P}^L(D_l)\,,\ \text{or}\ \tilde{P}^R(D_l)\ \text{or}\ \tilde{P}^V(D_l)\,. \tag{7.13}$$

Consider the Lax operator for bundles with the characteristic class defined by ϖ_n in the GS-basis (7.8), (7.10) $L = \tilde{L}_0 + L'_0 + L_1$. Here $\tilde{L}_0$ is the Lax operator of $D_{\frac{n}{2}}$,

$$L'_0(z) = \sum_{j \neq k}^{n} S'_{k,n+j} \mathbf{e}\,(\frac{2n-j-k}{2n-2} z) \phi(\frac{2n-j-k}{2n-2} \tau - u_j - u_k, z) \mathfrak{t}^0_{j,n+k}\,,$$

$$L_1(z) = \sum_{j=1}^{n} S^1_j \phi(\frac{1}{2}, z) \mathfrak{h}^1_j + \sum_{j \neq k}^{n} S^1_{j,k} \mathbf{e}\,(\frac{k-j}{2n-2} z) \phi(\frac{k-j}{2n-2} \tau - u_j + u_k + \frac{1}{2}, z) \mathfrak{t}^1_{k,j}$$

$$- \sum_{j \neq k}^{n} S^1_{j,n+k} \mathbf{e}\,(\frac{2n-j-k}{2n-2} z) \phi(\frac{2n-j-k}{2n-2} \tau - u_j - u_k - \frac{1}{2}, z) \mathfrak{t}^1_{k,n+j}\,.$$

From (7.9) and (7.11) after the diagonal reduction we come to the Hamiltonians

$$H'_0 = -\frac{1}{2} \sum_{j \neq k}^{n} S^1_{j,n+k} S^1_{k,n+j} E_2(\frac{2n-j-k}{2n-2} \tau - u_j - u_k)\,,$$

$$H_1 = -\frac{1}{2} \sum_{j=1}^{n} (S^1_j)^2 E_2(\frac{1}{2}) - \frac{1}{2} \sum_{j \neq k}^{n} S^1_{j,k} S^1_{k,j} E_2(\frac{k-j}{2n-2} \tau - u_j + u_k - \frac{1}{2})$$

$$+ \frac{1}{2} \sum_{j \neq k}^{n} S^1_{j,n+k} S^1_{n+k,j} E_2(\frac{2n-j-k}{2n-2} \tau - u_j - u_k - \frac{1}{2})\,.$$

Note that in all expressions $u_j = -u_{n+1-j}$.

Summarizing, the Hamiltonian $H^{CM}_{D_l} + H'_0 + H_1$ describes the integrable systems corresponding to the bundles with characteristic classes defined by ϖ_n, or ϖ_{n-1} with the moduli spaces (7.12) for l odd, or (7.13) for l even.

8 $\mathbf{E}_6$.

Roots and weights

The Cartan subalgebra of the Lie algebra $\mathbf{e_6}$ is the space

$$\mathfrak{H}(\mathbf{e_6}) = \{\mathbf{u} \in \mathbb{C}^7 \,|\, u_5 + u_6 + u_7 = 0\}\,. \tag{8.1}$$

The root system $R(\mathbf{e_6})$ is related to the root system $R(\mathbf{so(8)})$ of $\mathbf{so(8)}$

$$R(\mathbf{so(8)}) = \sum_{j \neq k}^{4} \pm e_j \pm e_k\,,\ \ \sharp\,(R(\mathbf{so(8)})) = 24\,.$$

Let

$$P^L = \{\varpi^L_a\}\,,\ P^R = \{\varpi^R_a\}\,,\ P^V = \{\varpi^V_a\}\,,\ \ (a = 1, \ldots, 8)$$

be the weights of the left (L) and right (R) spinor representations $\underline{8}^L$, $\underline{8}^R$ and the vector representation (V) $\underline{8}^V$ of $\mathbf{so(8)}$. They are equal to the following combinations of the basic vectors

$$\begin{array}{ll} \varpi^L_a \to \frac{1}{2} \sum_{k=1}^{4} \pm e_k\,, & \text{even number of negative terms}\,, \\ \varpi^R_a \to \frac{1}{2} \sum_{k=1}^{4} \pm e_k\,, & \text{odd number of negative terms}\,, \\ \varpi^V_a \to \pm e_k\,. & \end{array} \tag{8.2}$$

In these terms

$$R(\mathbf{e_6}) = R(\mathbf{so(8)}) \cup (P^L \pm \frac{1}{\sqrt{2}}(e_5 - e_7)) \cup (P^R \pm \frac{1}{\sqrt{2}}(e_5 - e_6)) \cup (P^V \pm \frac{1}{\sqrt{2}}(e_6 - e_7)) = \quad (8.3)$$

$$R(\mathbf{so(8)}) \cup \{\alpha^L_{a,\pm} = (\varpi^L_a \pm \frac{1}{\sqrt{2}}(e_5 - e_7))\,,\alpha^R_{a,\pm} = (\varpi^R_a \pm \frac{1}{\sqrt{2}}(e_5 - e_6))\,,\alpha^V_{a,\pm} = (\varpi^V_a \pm \frac{1}{\sqrt{2}}(e_6 - e_7))\}\,.$$

The systems of $\mathbf{e_6}$ roots is self-dual and the corresponding Dynkin diagram is simply-laced.

Since $\sharp(P^L) = \sharp(P^R) = \sharp(P^V) = 16$ the number of roots is equal $\sharp(R(\mathbf{e_6})) = 24+16+16+16 = 72$. We have from here $\dim \mathbf{e_6} = \mathrm{rank}\,\mathbf{e_6} + \sharp(R(\mathbf{e_6})) = 78$.

It follows from here that $\mathbf{so(8)}$ is subalgebra of $\mathbf{e_6}$. It can be found that

$$\mathbf{e_6} = \mathbf{so(8)} \oplus \mathcal{J} = \underline{28} + \underline{50}\,, \tag{8.4}$$

where $\mathcal{J}$ is a representation space of $\mathbf{so(8)}$. It is decomposed on irreducible components as

$$\mathcal{J} = 2 \times \underline{1} + 2 \times \underline{8}^L + 2 \times \underline{8}^R + 2 \times \underline{8}^R\,. \tag{8.5}$$

Here two scalar representations complete the Cartan subalgebra $\mathfrak{H}(\mathbf{so(8)})$ to the Cartan subalgebra $\mathfrak{H}(\mathbf{e_6})$ (8.1).

The basis of simple roots can be chosen as

$$\Pi = \begin{cases} \alpha_1 = \frac{1}{2}(e_4 - e_3 - e_2 - e_1) + \frac{1}{\sqrt{2}}(e_5 - e_6)\,, \\ \alpha_2 = e_3 - e_4\,, \\ \alpha_3 = e_2 - e_3\,, \\ \alpha_4 = e_1 - e_2\,, \\ \alpha_5 = -e_1 + \frac{1}{\sqrt{2}}(e_6 - e_7)\,, \\ \alpha_6 = e_4 + e_3\,. \end{cases} \tag{8.6}$$

The subsystem of simple roots

$$\Pi_1 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\} \tag{8.7}$$

is a system of simple roots of subalgebra $\mathbf{so(8)}$.

The Weyl group $W_{\mathbf{e_6}}$ of $R(\mathbf{e_6})$ is generated by $W_{\mathbf{so(8)}}$ (5.3) and by the reflections s_{α_1}, s_{α_5}

$$W_{\mathbf{e_6}} = \{W_{\mathbf{so(8)}}\,,\ (e_1 \leftrightarrow -e_1\,, e_6 \leftrightarrow e_7)\,,\ (e_j \leftrightarrow -e_j\,,\, j = 1, \dots, 6)\}\,. \tag{8.8}$$

The defined below λ_1 (8.17) is an element from $W_{\mathbf{e_6}}$

The simple roots (8.6) define the fundamental Weyl chamber

$$C = \{\mathbf{u} \in \mathfrak{H} \,|\, u_1 > u_2 > u_3 > u_4 > 0\,,\ u_5 - u_6 > \sqrt{2}\sum_{j=1}^{4} u_j\,,\ u_6 - u_7 > \sqrt{2}u_1\}\,.$$

The minimal root is

$$\alpha_0 = -\frac{1}{2}(e_4 + e_3 + e_2 + e_1) - \frac{1}{\sqrt{2}}(e_5 - e_7) \tag{8.9}$$
$$= -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6)\,.$$

Then

$$C_{alc} = \{\mathbf{u} \in \mathfrak{H} \mid u_1 > u_2 > u_3 > u_4 > 0\,, \tag{8.10}$$

$$u_5-u_6>\sqrt{2}\sum_{j=1}^{4}u_j\,,\;u_6-u_7>\sqrt{2}u_1\,,\;\frac{1}{2}\sum_{j=1}^{4}u_j+\sqrt{2}(u_5-u_7)<1\}\,.$$

The subset of positive roots corresponding to $\Pi_{\mathbf{e_6}}$ assumes the form

$$R^+(\mathbf{e_6})=R^+(\mathbf{so(8)})\cup(P^L+\frac{1}{\sqrt{2}}(e_5-e_7))\cup(P^R+\frac{1}{\sqrt{2}}(e_5-e_6))\cup(P^V+\frac{1}{\sqrt{2}}(e_6-e_7))\,. \quad (8.11)$$

This data allows one to construct the Chevalley basis in $\mathbf{e_6}$.

It follows from (8.6) that the root lattice of $\mathbf{e_6}$ is

$$Q(\mathbf{e_6})=Q(\mathbf{so(8)})+\mathbb{Z}(-e_1+\frac{1}{\sqrt{2}}(e_6-e_7))+\mathbb{Z}(\frac{1}{2}(e_4-e_3-e_2-e_1)+\frac{1}{\sqrt{2}}(e_5-e_6))\,. \quad (8.12)$$

The fundamental weights dual to $\Pi_{\mathbf{e_6}}$ (8.6) are

$$\left\{\begin{array}{l}
\varpi_1=\frac{\sqrt{2}}{3}(2e_5-e_6-e_7)=\frac{1}{3}(4\alpha_1+5\alpha_2+6\alpha_3+4\alpha_4+2\alpha_5+3\alpha_6),\,,\\
\varpi_2=\frac{1}{2}(e_1+e_2+e_3-e_4)+\frac{1}{3\sqrt{2}}(5e_5-e_6-4e_7)=\frac{1}{3}(5\alpha_1+10\alpha_2+12\alpha_3+8\alpha_4+4\alpha_5+6\alpha_6)\,,\\
\varpi_3=e_1+e_2+\sqrt{2}(e_5-e_7)=2\alpha_1+4\alpha_2+6\alpha_3+4\alpha_4+2\alpha_5+3\alpha_6\,,\\
\varpi_4=e_1+\frac{1}{3\sqrt{2}}(4e_5+e_6-5e_7)=\frac{1}{3}(4\alpha_1+8\alpha_2+12\alpha_3+10\alpha_4+5\alpha_5+6\alpha_6)\,,\\
\varpi_5=\frac{\sqrt{2}}{3}(e_5+e_6-2e_7)=\frac{1}{3}(2\alpha_1+4\alpha_2+6\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6)\,,\\
\varpi_6=\frac{1}{2}(e_1+e_2+e_3+e_4)+\frac{1}{\sqrt{2}}(e_5-e_7)=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\alpha_6\,.
\end{array}\right. \quad (8.13)$$

The vector $\rho_{\mathbf{e_6}}$ takes the form $\rho_{\mathbf{e_6}}=\rho_{\mathbf{so(8)}}+\frac{8}{\sqrt{2}}(e_5-e_7)=3e_1+2e_2+e_3+\frac{8}{\sqrt{2}}(e_5-e_7)$ and since $h=12$

$$\kappa_{\mathbf{e_6}}=\frac{1}{12}\rho_{\mathbf{so(8)}}+\frac{2}{3\sqrt{2}}(e_5-e_7)=\frac{1}{4}e_1+\frac{1}{6}e_2+\frac{1}{12}e_3+\frac{2}{3\sqrt{2}}(e_5-e_7)\,. \quad (8.14)$$

The weight lattice $P(\mathbf{e_6})$ is generated by the root lattice $Q(\mathbf{e_6})$ (8.15) and one of the fundamental weights ϖ_k $(k=1,2,4,5)$

$$P(\mathbf{e_6})=Q(\mathbf{e_6})+\mathbb{Z}\varpi_1\,. \quad (8.15)$$

The factor group P/Q being isomorphic to the center of the universal covering group E_6 is

$$P(\mathbf{e_6})/Q(\mathbf{e_6})\sim\mu_3\,. \quad (8.16)$$

It follows from (8.9) that the fundamental alcove (8.10) has the vertices

$$C_{alc}=(0,\varpi_1,\frac{1}{2}\varpi_2,\frac{1}{3}\varpi_3,\frac{1}{2}\varpi_4,\varpi_5,\frac{1}{2}\varpi_6)\,.$$

The transformation $\lambda_1\in\Gamma_{C_{alc}}$ generated by ϖ_1 acts on the extended Dynkin diagram as

$$\lambda_1=\left\{\begin{array}{ll}
\alpha_1\to\alpha_0\,, & e_1\to\frac{1}{2}(e_1+e_2+e_3-e_4)\,,\\
\alpha_2\to\alpha_6\,, & e_2\to\frac{1}{2}(e_1+e_2-e_3+e_4)\,,\\
\alpha_3\to\alpha_3\,, & e_3\to\frac{1}{2}(e_1-e_2+e_3+e_4)\,,\\
\alpha_4\to\alpha_2\,, & e_4\to\frac{1}{2}(e_1-e_2-e_3-e_4)\,,\\
\alpha_5\to\alpha_1\,, & e_5\to e_6\,,\\
\alpha_6\to\alpha_4\,, & e_6\to e_7\,,\\
\alpha_0\to\alpha_5\,, & e_7\to e_5\,.
\end{array}\right. \quad (8.17)$$

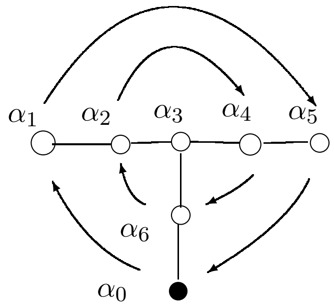


Fig.7 $\mathbf{e}_6$, λ_1

Note that being restricted on $\mathbf{so(8)}$ λ acts on the fundamental weights as

$$\varpi^V \to \varpi^R \to \varpi^L \, .$$

Chevalley basis

We start with the Chevalley basis in subalgebra $\mathbf{so(8)}$. It is generated by the canonical basis (e_1, e_2, e_3, e_4) in $\mathfrak{H}(\mathbf{so(8)})$ and the root subspaces

$$\begin{array}{ll} \alpha_{jk} = (e_j - e_k) \to E_{j,k} \, , \quad j \neq k \, , & \langle \kappa_{\mathbf{e_6}}, \alpha_{jk} \rangle = \frac{j-k}{12} \, , \\ \alpha_{j,k+4} = (e_j + e_k) \to E_{j,k+4} \, , \quad j \neq k \, , & \langle \kappa_{\mathbf{e_6}}, \alpha_{jk} \rangle = \frac{j+k-8}{12} \, , \\ \alpha_{j+4,k} = (-e_j - e_k) \to E_{j+4,k} \, , \quad j \neq k, k+4 \, , \quad j \neq k \, , & \langle \kappa_{\mathbf{e_6}}, \alpha_{jk} \rangle = \frac{-j-k+8}{12} \, , \, . \end{array} \tag{8.18}$$

The root subspaces in $R(\mathbf{e_6}) \setminus R(\mathbf{so(8)})$ are representations $2 \times \underline{8}^L + 2 \times \underline{8}^R + 2 \times \underline{8}^R$ of $\mathbf{so(8)}$ (8.5)

$$\alpha^L_{a,\pm} = (\varpi^L_a \pm \frac{1}{\sqrt{2}}(e_5 - e_7)) \to E^L_{a,\pm} \, , \quad a = 1, \dots, 8 \, , \quad \langle \kappa_{\mathbf{e_6}}, \alpha^L_{a,\pm} \rangle = \pm \frac{2}{3} + \langle \kappa_{\mathbf{so(8)}}, \varpi^L_a \rangle \, ,$$

$$\alpha^R_{a,\pm} = (\varpi^R_a \pm \frac{1}{\sqrt{2}}(e_5 - e_6)) \to E^R_{a,\pm} \, , \quad a = 1, \dots, 8 \, , \quad \langle \kappa_{\mathbf{e_6}}, \alpha^R_{a,\pm} \rangle = \pm \frac{1}{3} + \langle \kappa_{\mathbf{so(8)}}, \varpi^R_a \rangle \, , \tag{8.19}$$

$$\alpha^V_{a,\pm} = (\varpi^V_a \pm \frac{1}{\sqrt{2}}(e_6 - e_7)) \to E^V_{a,\pm} \, , \quad a = 1, \dots, 8 \quad \langle \kappa_{\mathbf{e_6}}, \alpha^V_{a,\pm} \rangle = \pm \frac{1}{3} + \langle \kappa_{\mathbf{so(8)}}, \varpi^V_a \rangle \, .$$

It follows from (8.2) that $-\varpi^{L,R,V}_a$ is again a spinor weight of the same type. We denote by $E^A_{a,\pm}$ the Chevalley generator related to ϖ^A_a.

The Killing form in the Chevalley basis assumes the form

$$(e_j, e_k) = \delta_{j,k} \, , \quad (E_{j,k}, E_{l,m}) = \delta_{k,l}\delta_{j,m} \, , \quad (j, k = 1, \dots, 4) \, ,$$

$$(E^A_{a,\pm}, E^B_{b,\mp}) = \delta_{a,-b}\delta^{A,B} \, , \quad A, B = L, R, V \, . \tag{8.20}$$

GS basis

Consider the Weyl action λ_1 (8.17) on the root spaces of $\mathbf{e_6}$. It easy to see that λ_1 preserves $\mathbf{so(8)}$ and therefore $\mathcal{J}$ in (8.4).

GS basis in $\mathbf{so(8)}$.

Consider the action of λ_1 on $\mathbf{so(8)}$. Note first, that an orbit of λ_1 that does not contains the minimal root is Π_1 (8.7). It is a basis in the Cartan subalgebra $\mathfrak{H}(\mathbf{so(8)})$. Since $\lambda_1^3 = 1$

$$\mathfrak{H}(\mathbf{so(8)}) = \tilde{\mathfrak{H}}_0 + \mathfrak{H}_1 + \mathfrak{H}_2 \,, \quad \lambda_1 \mathfrak{H}_j = \omega^j \mathfrak{H}_j \,, \quad \omega = \exp \frac{2\pi i}{3} \,. \tag{8.21}$$

Here $\tilde{\mathfrak{H}}_0$ is a Cartan subalgebra in $\tilde{\mathfrak{g}}_0$. The basis $\tilde{\Pi}^\vee$ of simple coroots is

$$\tilde{\Pi}^\vee = \{\tilde{\alpha}_3^\vee = \sum_{k=0}^{2} \lambda^k \alpha_3 = 3\alpha_3 = 3(e_2 - e_3)\,, \;\; \tilde{\alpha}_2^\vee = \sum_{k=0}^{2} \lambda^k \alpha_2 = (e_1 - e_2 + 2e_3)\} \,. \tag{8.22}$$

The dual system is a system of simple roots of $\mathbf{g_2}$

$$\tilde{\Pi} = \{\tilde{\alpha}_3 = \frac{1}{3}(e_2 - e_3)\,, \; \tilde{\alpha}_2 = \frac{1}{3}(e_1 - e_2 + 2e_3)\} \,. \tag{8.23}$$

In this basis the positive $\mathbf{g_2}$ roots are

$$\tilde{R}^+(\mathbf{g_2}) = (\tilde{\Pi}\,, \; \frac{1}{3}(2,1,1,0)\,, \; \frac{1}{3}(1,2,-1,0)\,, \; \frac{1}{3}(1,1,0,0)\,, \; \frac{1}{3}(1,0,1,0)) \,. \tag{8.24}$$

It is convenient to pass from the coroot basis $\tilde{\Pi}^\vee$ in $\mathfrak{H}(\mathbf{g_2})$ to the canonical basis (e_1, e_2, e_3)

$$\tilde{\mathbf{u}} = u_1 e_1 + u_2 e_2 + u_3 e_3 \,, \quad u_1 = u_2 + u_3 \,, \quad (u_4 = 0) \,. \tag{8.25}$$

According with (8.21) the GS basis in $\mathfrak{H}(\mathbf{so(8)})$ is generated by (8.25) and by

$$\left\{ \begin{array}{l} \mathfrak{h}^1_{\alpha_2} = \{\frac{1}{\sqrt{3}}(\alpha_2 + \omega\alpha_4 + \omega^2\alpha_6)\} = \{\frac{1}{\sqrt{3}}(\omega, -\omega, 1+\omega^2, -1+\omega^2)\}\,, \\ \mathfrak{h}^2_{\alpha_2} = \{\frac{1}{\sqrt{3}}(\alpha_2 + \omega^2\alpha_4 + \omega\alpha_6)\} = \{\frac{1}{\sqrt{3}}(\omega^2, -\omega^2, 1+\omega, -1+\omega)\} \,. \end{array} \right. \tag{8.26}$$

With respect to the λ_1 $\mathbf{so(8)}$ is decomposed as

$$\mathbf{so(8)} = \mathbf{g}_2 \oplus \underline{7} \oplus \underline{7}'\,, \quad \lambda_1(\mathbf{g}_2) = \mathbf{g}_2\,, \; \lambda_1(\underline{7}) = \omega \cdot \underline{7}\,, \; \lambda_1(\underline{7}') = \omega^2 \cdot \underline{7}'\,, \tag{8.27}$$

where $\underline{7}$, $\underline{7}'$ are fundamental representations of $\mathbf{g}_2$.

According with the gradation we have the following gradation of the root spaces

$\mathbf{g_2}$	$\underline{7}$	$\underline{7}'$
$E^0_{12} = E_{12} + E_{34} + E_{35}$	$\mathfrak{t}^1_{12} = \frac{1}{\sqrt{3}}(E_{12} + \omega E_{34} + \omega^2 E_{35})$	$\mathfrak{t}^2_{12} = \frac{1}{\sqrt{3}}(E_{12} + \omega^2 E_{34} + \omega E_{35})$
$E^0_{13} = E_{13} + E_{24} + E_{25}$	$\mathfrak{t}^1_{13} = \frac{1}{\sqrt{3}}(E_{13} + \omega E_{24} + \omega^2 E_{25})$	$\mathfrak{t}^2_{13} = \frac{1}{\sqrt{3}}(E_{13} + \omega^2 E_{24} + \omega E_{25})$
$E^0_{14} = E_{14} + E_{15} + E_{26}$	$\mathfrak{t}^1_{14} = \frac{1}{\sqrt{3}}(E_{14} + \omega E_{15} + \omega^2 E_{26})$	$\mathfrak{t}^2_{14} = \frac{1}{\sqrt{3}}(E_{14} + \omega^2 E_{15} + \omega E_{26})$
$E^0_{23} = E_{23}\,,\; E^0_{16} = E_{16}\,,$ $E^0_{17} = E_{17}$	$\mathfrak{t}^1_{(2,3,4),1}\,,\; \mathfrak{h}^1_{\alpha_2}$	$\mathfrak{t}^2_{(2,3,4),1}\,,\; \mathfrak{h}^2_{\alpha_2}$

Table1. GS basis in $\mathbf{so(8)}$.

The roots of $\mathbf{so(8)}$ that parameterized the GS basis are

$$(e_1 - e_m) \to \mathfrak{t}^a_{1,m}\,, \quad (e_m - e_1) \to \mathfrak{t}^a_{m,1}\,.$$

The left column of table 5 contains seven positive root subspaces of $\mathbf{g_2}$. In particular, $E^0_{23} = \tilde{E}_{\alpha_3}$ and $E^0_{12} = \tilde{E}_{\alpha_2}$.

The Killing form on $\mathbf{g}_2$ is the canonical form on $\mathfrak{H}(\mathbf{g}_2)$, and for the root subspaces non-vanishing elements are

$$(E^0_{12}, E^0_{21}) = (E^0_{13}, E^0_{31}) = (E^0_{14}, E^0_{41}) = 3\,, \tag{8.28}$$
$$(E^0_{23}, E^0_{32}) = (E^0_{16}, E^0_{61}) = (E^0_{17}, E^0_{71}) = 1\,.$$

For rest generators of the GS basis in $\mathbf{so(8)}$ we have

$$(\mathfrak{h}^1_{\alpha_2}, \mathfrak{h}^2_{\alpha_2}) = 2\,, \quad (\mathfrak{t}^{k_1}_{jk}, \mathfrak{t}^{k_2}_{lm}) = \delta^{(k_1+k_2,0)}\delta_{j,m}\delta_{k,l}\,. \tag{8.29}$$

GS basis in $\mathcal{J}$.

The basis in the Cartan part of $\mathcal{J}$

$$\mathfrak{H}(\mathcal{J}) = \{u_5 e_5 + u_6 e_6 + u_7 e_7\,, \quad u_5 + u_6 + u_7 = 0\}\,.$$

is transformed as $\lambda_1(e_5) = e_7\,,\ \lambda_1(e_6) = e_5\,,\ \lambda_1(e_7) = e_6$.

The basis $E^A_{a,\pm}$ (8.19) is transformed as

$$\lambda_1\ :\ \ E^L_{a,+} \to E^V_{a,-} \to E^R_{a,-}\,, \quad E^L_{a,-} \to E^V_{a,+} \to E^R_{a,+}\,.$$

There are 16 orbits of this type in $\mathcal{J}$. Thus, the GS basis in $\mathcal{J}$ takes the form

$$\mathfrak{t}^k_{a,+} = \frac{1}{\sqrt{3}}(E^L_{a,+} + \omega^k E^V_{a,-} + \omega^{2k} E^R_{a,-})\,, \tag{8.30}$$
$$\mathfrak{t}^k_{a,-} = \frac{1}{\sqrt{3}}(E^L_{a,-} + \omega^k E^V_{a,+} + \omega^{2k} E^R_{a,+})\,, \quad (k = 0,1,2)\,,$$
$$\mathfrak{h}^k_5 = \frac{1}{\sqrt{3}}(e_5 + \omega^k e_6 + \omega^{2k} e_7)\,, \quad (k = 1,2)\,.$$

Here $\mathfrak{t}^0_{a,5,7}$, $\mathfrak{t}^0_{a,7,5}$ generate the GS basis in V (5.29.I).

The Killing form is

$$(\mathfrak{t}^{k_1}_{a,+}, \mathfrak{t}^{k_2}_{b,-}) = \delta_{a,-b}\delta^{(k_1+k_2,0)}\,, \quad (\mathfrak{h}^{k_1}_5, \mathfrak{h}^{k_2}_5) = \delta^{(k_1+k_2,0)}\,. \tag{8.31}$$

In summary, under the λ_1 action $\mathbf{e_6}$ is decomposed as

$$\mathbf{e_6} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2\,, \tag{8.32}$$

where

$$\mathfrak{g}_0 = \mathbf{g_2} + V\,, \quad V = \{\mathfrak{t}^0_{a,+}\,,\ \mathfrak{t}^0_{a,-}\}\,, \quad \dim V = 16\,.$$

The λ_1-invariant subalgebra $\mathfrak{g}_0$ is isomorphic to $\mathbf{so(8)}+2\times\underline{1}$. It is obtained from the $\mathbf{e_6}$ extended Dynkin diagram (Fig.7) by dropping out three vertices α_0, α_1 and α_5 .

The subspaces $\mathfrak{g}_1$ and $\mathfrak{g}_2$ have the GS basis of the form

$$\mathfrak{g}_1 = \left\{\mathfrak{t}^1_{a,+}\,,\ \mathfrak{t}^1_{a,-}\,,\ \mathfrak{h}^1_5\ \ \mathfrak{h}^1_{\alpha_2}\,,\ \mathfrak{t}^1_{1,(2,3,4)}\,,\ \mathfrak{t}^1_{(2,3,4),1}\right\}\,, \tag{8.33}$$

$$\mathfrak{g}_2 = \{\mathfrak{t}^2_{a,+}\,,\ \mathfrak{t}^2_{a,-}\,,\ \mathfrak{h}^2_5\,,\ \mathfrak{h}^2_{\alpha_2}\,,\ \mathfrak{t}^2_{1,(2,3,4)}\,,\ \mathfrak{t}^2_{(2,3,4),1}\}\,.$$

In correspondence with (8.32) we have

$$\underline{78} = \underline{30} + \underline{24} + \underline{24}\,.$$

The form of the dual basis (5.9.I), (5.15.I) follows from (8.29), (8.31)

$$\mathfrak{H}^k_{\alpha_2} = \frac{1}{2}\mathfrak{h}^{3-k}_{\alpha_2}\,,\quad \mathfrak{T}^k_{jm} = \mathfrak{t}^{3-k}_{mj}\,,\quad (k=1,2)\,, \tag{8.34}$$

$$\mathfrak{T}^k_{a,+} = \mathfrak{t}^{3-k}_{-a,-}\,,\quad \mathfrak{T}^k_{a,-} = \mathfrak{t}^{3-k}_{-a,+}\,,\quad (k=1,2)\,,$$

$$\mathfrak{H}^k_5 = \mathfrak{h}^{3-k}_5\,,\quad (k=1,2)\,.$$

Lax operators and Hamiltonians

Trivial bundles

The moduli space of trivial $\bar{G} = E_6$ bundles is the quotient

$$\mathfrak{H}_{\mathbf{e_6}}/(W_{\mathbf{e_6}} \ltimes (\tau Q(\mathbf{e_6}) + Q(\mathbf{e_6}))\,,$$

where $W_{\mathbf{e_6}}$ is defined in (8.8) It means in particular that $\mathbf{u} \in \mathfrak{H}_{\mathbf{e_6}}$ is defined up to the shifts

$$\mathbf{u} \sim \mathbf{u} + \tau\gamma_1 + \gamma_2\,,\quad \gamma_{1,2} \in Q(\mathbf{e_6})\,. \tag{8.35}$$

where $Q(\mathbf{e_6})$ is (8.12).

The moduli space of trivial $G^{ad} = E_6/\mu_3$ bundles is the quotient

$$\mathfrak{H}_{\mathbf{e_6}}/(W_{\mathbf{e_6}} \ltimes (\tau P(\mathbf{e_6}) + P(\mathbf{e_6})))\,.$$

It means that

$$\mathbf{u} \sim \mathbf{u} + \tau\gamma_1 + \gamma_2\,,\quad \gamma_{1,2} \in P(\mathbf{e_6})\,, \tag{8.36}$$

and $P(\mathbf{e_6})$ is (8.16).

For trivial bundles the Lax operator takes the form

$$L^{CM}_{e_6}(z) = L^{CM}_{so(8)} + \sum_{j=5}^{7}(v_j + S_{0,j}E_1(z))e_j + \tag{8.37}$$

$$\sum_{a=1}^{8}\Big(S^L_{a,+}\phi((\mathbf{u},\varpi^L_a) + \frac{1}{\sqrt{2}}(u_5-u_7),z)E^L_{a,+} + S^L_{a,-}\phi((\mathbf{u},\varpi^L_a) + \frac{1}{\sqrt{2}}(u_7-u_5),z)E^L_{a,-} +$$

$$S^R_{a,+}\phi((\mathbf{u},\varpi^R_a) + \frac{1}{\sqrt{2}}(u_5-u_6),z)E^R_{a,+} + S^R_{a,-}\phi((\mathbf{u},\varpi^R_a) + \frac{1}{\sqrt{2}}(u_6-u_5),z)E^R_{a,-} +$$

$$S^V_{a,+}\phi((\mathbf{u},\varpi^V_a) + \frac{1}{\sqrt{2}}(u_6-u_7),z)E^V_{a,+} + S^V_{a,-}\phi((\mathbf{u},\varpi^V_a) + \frac{1}{\sqrt{2}}(u_7-u_6),z)E^V_{a,-}\Big)\,.$$

After symplectic reduction with respect to the action of the Cartan subgroup we come to integrable E_6 hierarchy with two types of moduli space (8.35), (8.36). The quadratic Hamiltonian is

$$H^{CM}_{e_6} = H^{CM}_{so(8)} + \frac{1}{2}\sum_{j=5}^{7}v_j^2 + \sum_{a=1}^{8}\Big(S^L_{a,+}S^L_{-a,-}E_2((\mathbf{u},\varpi^L_a) + \frac{1}{\sqrt{2}}(u_5-u_6)) \tag{8.38}$$

$$S^R_{a,+}S^R_{-a,-}E_2((\mathbf{u},\varpi^R_a) + \frac{1}{\sqrt{2}}(u_5-u_6)) + S^V_{a,+}S^V_{-a,-}E_2((\mathbf{u},\varpi^V_a) + \frac{1}{\sqrt{2}}(u_6-u_7))\Big)\,.$$

Nontrivial bundles

The moduli space.
Now $\mathbf{u}=\tilde{\mathbf{u}}\in\mathfrak{H}(\mathbf{g_2})$

$$\tilde{\mathbf{u}}=u_1e_1+u_2e_2+u_3e_3\,,\quad u_1=u_2+u_3\,. \tag{8.39}$$

More exactly, $\tilde{\mathbf{u}}$ belongs to fundamental domains with respect to action of affine Weyl group corresponding to $\mathbf{g_2}$. The Weyl group $W_{\mathbf{g_2}}$ is isomorphic to the dihedral group of order 12 with two generators

$$(u_1,u_2,u_3)\to(u_1,u_3,u_2)\,,$$

$$\left(\begin{array}{c}u_1\\u_2\\u_3\end{array}\right)\to\frac{1}{3}\left(\begin{array}{rrr}2&1&-2\\1&2&2\\-2&2&-1\end{array}\right)\left(\begin{array}{c}u_1\\u_2\\u_3\end{array}\right).$$

The invariant root sublattice $\tilde{Q}^\vee=Q^\vee(\mathbf{g_2})$ coincides with $\tilde{P}^\vee=P^\vee(\mathbf{g_2})$. From (8.22) we find

$$\tilde{Q}^\vee(\mathbf{g_2})=\{m_2e_1+(3m_3-m_2)e_2+(2m_2-3m_3)e_3\,|\,m_2,m_3\in\mathbb{Z}\}\,.$$

Then the moduli space of nontrivial E_6 bundle is defined as

$$\mathfrak{H}(\mathbf{g_2})/(W_{\mathbf{g_2}}\ltimes(\tau\tilde{Q}^\vee(\mathbf{g_2})+\tilde{Q}^\vee(\mathbf{g_2})))\,. \tag{8.40}$$

In other words

$$u_1\sim u_1+\tau m_2+m_2'\,,\quad u_2\sim u_2+\tau(3m_3-m_2)+3m_3'-m_2'\,,$$

$$u_3\sim u_3+\tau(2m_2-3m_3)+2m_2'-3m_3'\,,\quad m_{2,3}\in\mathbb{Z}\,.$$

The Lax operators
According with (8.32) $L(z)=\tilde{L}_0(z)+L_0'(z)+L_1(z)+L_2(z)$. Here $\tilde{L}_0(z)=L^{CM}_{\mathbf{g_2}}(z)$. Let $\tilde{\mathbf{v}}=(v_1,v_2,v_3)$ $(v_1=v_2+v_3)$ be the Poisson dual vector to $\tilde{\mathbf{u}}$ (8.39). Following to (8.24) and table 5 we find

$$L^{CM}_{\mathbf{g_2}}(z)=\sum_{j=1}^{3}(v_j+S_{0,j}E_1(z))e_j+S^0_{12}\phi(\frac{1}{3}(u_1-u_2+2u_3),z)E^0_{12}+S^0_{21}\phi(\frac{1}{3}(-u_1+u_2-2u_3),z)E^0_{21}$$

$$+S^0_{23}\phi(\frac{1}{3}(u_2-u_3),z)E^0_{23}+S^0_{32}\phi(\frac{1}{3}(u_3-u_2),z)E^0_{32}+S^0_{16}\phi(\frac{1}{3}(u_1+u_3),z)E^0_{16}+S^0_{61}\phi(\frac{1}{3}(-u_1-u_3),z)E^0_{61}$$

$$+S^0_{17}\phi(\frac{1}{3}(u_1+u_2),z)E^0_{17}+S^0_{71}\phi(\frac{1}{3}(-u_1-u_2),z)E^0_{71}+S^0_{14}\phi(\frac{1}{3}(2u_1+u_2+u_3),z)E^0_{14}$$

$$+S^0_{41}\phi(\frac{1}{3}(-2u_1-u_2-u_3),z)E^0_{41}+S^0_{13}\phi(\frac{1}{3}(u_1+2u_2-u_3),z)E^0_{13}+S^0_{31}\phi(\frac{1}{3}(-u_1-2u_2+u_3),z)E^0_{31}\,.$$

Define φ^k_β (6.14.I)

$$\varphi^k_\beta(\tilde{\mathbf{u}},z)=\mathbf{e}\left(\langle\kappa_{\mathbf{e_6}},\beta\rangle z\right)\phi(\langle\tilde{\mathbf{u}}+\kappa_{\mathbf{e_6}}\tau,\beta\rangle+k/3,z)\,,\quad(k=0,1,2)\,,$$

where $\beta=\alpha_{jk}$ and $\langle\kappa_{\mathbf{e_6}},\beta\rangle$ (8.18), or $\beta=\alpha^L_{(\pm a,\pm)}$ and $\langle\kappa_{\mathbf{e_6}},\beta\rangle$ (8.19).

Then following (6.15.I) and (6.17.I) we find

$$L'_0(z) = \sum_{a=1}^{8}\Big(S^0_{a,-}\varphi^0_{\alpha^L_{(-a,-)}}(-\tilde{\mathbf{u}},z)\mathfrak{t}^0_{a,+} + S^0_{a,+}\varphi^0_{\alpha^L_{(a,+)}}(-\tilde{\mathbf{u}},z)\mathfrak{t}^0_{a,-}\Big)\,,$$

and for $k = 1,2$

$$L_k(z) = S_5^{3-k}\phi(k/3,z)\mathfrak{h}_5^k + S_{\alpha_2}^{3-k}\phi(k/3,z)\mathfrak{h}^k_{\alpha_2} + \sum_{m=2}^{4}(S^{3-k}_{m,1}\varphi^k_{\alpha_{1,m}}(-\tilde{\mathbf{u}},z)\mathfrak{t}^k_{1,m} + S^{3-k}_{1,m}\varphi^k_{\alpha_{m,1}}(-\tilde{\mathbf{u}},z)\mathfrak{t}^k_{m,1}) +$$

$$\sum_{a=1}^{8}(S^{3-k}_{-a,-}\varphi^k_{\alpha^L_{(a,+)}}(-\tilde{\mathbf{u}},z)\mathfrak{t}^k_{a,+} + S^{3-k}_{-a,+}\varphi^k_{\alpha^L_{(a,-)}}(-\tilde{\mathbf{u}},z)\mathfrak{t}^k_{a,-})\,.$$

After the symplectic reduction we come to the integrable system with quadratic Hamiltonian $H_{\mathbf{e_6}} = H^{CM}_{\mathbf{g_2}} + H'_0 + H_1$, where H'_0 is defined by $\frac{1}{2}\mathrm{tr}(L_0^{'2})$, H_1 by $\mathrm{tr}(L_1L_2)$, and

$$H^{CM}_{\mathbf{g_2}} = \frac{1}{2}\sum_{j=1}^{3}v_j^2 - 3S^0_{12}S^0_{21}E_2(\frac{1}{3}(u_1-u_2+2u_3)) - S^0_{23}S^0_{32}E_2(\frac{1}{3}(u_2-u_3))$$

$$-S^0_{16}S^0_{61}E_2(\frac{1}{3}(u_1+u_3)) - S^0_{17}S^0_{71}E_2(\frac{1}{3}(u_1+u_2)) - 3S^0_{14}S^0_{41}E_2(\frac{1}{3}(2u_1+u_2+u_3))$$

$$-3S^0_{13}S^0_{31}E_2(\frac{1}{3}(u_1+2u_2-u_3))\,,$$

$$H'_0 = -\sum_{a=1}^{8}S^L_{a,+}S^L_{-a,-}E_2(\varpi^L_a,\tilde{\mathbf{u}})\,,$$

$$-H_1 = (S^1_5S^2_5 + 2S^1_{\alpha_2}S^2_{\alpha_2})E_2(1/3) + \sum_{m=2}^{4}(S^1_{m,1}S^2_{1,m} + S^1_{1,m}S^2_{m,1})E_2(u_1-u_m) +$$

$$\sum_{a=1}^{8}(S^1_{a,-}S^2_{-a,+} + S^1_{a,+}S^2_{-a,-})E_2(\varpi^L_a,\tilde{\mathbf{u}})\,.$$

9 $\mathbf{E}_7$.

Roots and weights

The Cartan subalgebra can be identified with the space $\mathfrak{H}_{\mathbf{e_7}} = \{\mathbf{u}\in\mathbb{C}^7\}$. The root system $R(\mathbf{e_7})$ is related to the root system $R(\mathbf{e_6})$ (8.3) as follows

$$R(\mathbf{e_7}) = R(\mathbf{e_6}) \cup \pm(\sqrt{2}e_{i+4}\,,\ i=1,2,3) \tag{9.1}$$

$$\cup(P^R \pm \frac{1}{\sqrt{2}}(e_5+e_6)) \cup (P^V \pm \frac{1}{\sqrt{2}}(e_6+e_7)) \cup (P^L \pm \frac{1}{\sqrt{2}}(e_5+e_7)) =$$

$$R(\mathbf{e_6}) \cup \{\alpha^+_{i+4}\,, \alpha^{V,+}_{a,\pm}\,, \alpha^{R,+}_{a,\pm}\,, \alpha^{L,+}_{a,\pm}\}\,,\ (a=1,\dots,8\,,\ i=1,2,3)$$

(compare with (8.3)). Then $\sharp(R(\mathbf{e_7})) = 72+6+3\times16 = 126$ and $\dim\,\mathbf{e_7} = \mathrm{rank}\,\mathbf{e_7} + \sharp(R(\mathbf{e_7})) = 133$.

The basis of simple roots can be chosen as

$$\Pi = \begin{cases} \alpha_1 = \frac{1}{2}(e_4 - e_3 - e_2 - e_1) + \frac{1}{\sqrt{2}}(e_5 - e_6)\,, \\ \alpha_2 = e_3 - e_4\,, \\ \alpha_3 = e_2 - e_3\,, \\ \alpha_4 = e_1 - e_2\,, \\ \alpha_5 = -e_1 + \frac{1}{\sqrt{2}}(e_6 - e_7)\,, \\ \alpha_6 = e_4 + e_3\,, \\ \alpha_7 = \sqrt{2}e_7\,. \end{cases} \tag{9.2}$$

The subsystem of simple roots

$$\Pi_1 = \{\alpha_i\,, \quad i = 1, \dots 6\} = \Pi(\mathbf{e_6}) \tag{9.3}$$

is a system of simple roots of subalgebra $\mathbf{e_6}$. It follows from (9.1) that the positive roots, corresponding to Π is

$$R^+(\mathbf{e_7}) = R^+(\mathbf{e_6}) \cup (\sqrt{2}e_{i+4}\,,\ i = 1, 2, 3)$$
$$\cup (P^R + \frac{1}{\sqrt{2}}(e_5 + e_6)) \cup (P^V + \frac{1}{\sqrt{2}}(e_6 + e_7)) \cup (P^L + \frac{1}{\sqrt{2}}(e_5 + e_7))$$

Then the root lattice takes the form

$$Q(\mathbf{e_7}) = Q(\mathbf{e_6}) + \mathbb{Z}\sqrt{2}e_7 + \mathbb{Z}(-e_1 + \frac{1}{\sqrt{2}}(e_6 + e_7)) + \mathbb{Z}(\frac{1}{2}(e_1 - e_2 - e_3 - e_4) + \frac{1}{\sqrt{2}}(e_5 + e_6))\,. \tag{9.4}$$

The simple roots (9.2) define the fundamental Weyl chamber $C = \{\mathbf{u} \in \mathfrak{H} \,|\, \langle \mathbf{u}, \alpha \rangle > 0\,,\ \alpha \in \Pi(\mathbf{e_7})\}$.

The minimal root is

$$\alpha_0 = -\sqrt{2}e_5 = -(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7)\,. \tag{9.5}$$

The root system $R_{\mathbf{e_7}}$ is self-dual and the corresponding Dynkin diagram is simply-laced.

The half-sum of the positive roots $\rho_{\mathbf{e_7}} = 3e_1 + 2e_2 + e_3 + \frac{1}{\sqrt{2}}(17e_5 + 9e_6 + e_7)$ can be expressed in terms of roots of subalgebras $\mathbf{e_6}$ or $\mathbf{f_4}$ (see below)

$$\rho_{\mathbf{e_7}} = \rho_{\mathbf{e_6}} + \frac{9}{\sqrt{2}}(e_5 + e_6 + e_7) = \rho_{\mathbf{f_4}} + \frac{1}{4}(3e_1 + e_2 + e_3 - e_4) + \frac{5}{2\sqrt{2}}(e_5 - e_7)\,,$$

where $\rho_{\mathbf{f_4}} = \frac{1}{4}(9, 7, 3, 1, \frac{11}{2\sqrt{2}}, 0, -\frac{11}{2\sqrt{2}})$. The Coxeter number is equal to $h = 18$. Then

$$\kappa_{\mathbf{e_7}} = \frac{1}{18}\kappa_{f_4} + \frac{1}{72}(3e_1 + e_2 + e_3 - e_4) + \frac{5}{36\sqrt{2}}(e_5 - e_7)\,, \quad (\kappa_{\mathbf{f_4}} = \frac{1}{18}\rho_{\mathbf{f_4}})\,. \tag{9.6}$$

We define the fundamental weights dual to Π (9.2)

$$\begin{array}{l} \varpi_1 = \sqrt{2}e_5 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7\,, \\ \varpi_2 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4) + \frac{1}{3\sqrt{2}}(5e_5 - e_6 - 4e_7) \\ = \frac{1}{2}(4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 9\alpha_4 + 6\alpha_5 + 7\alpha_6 + 3\alpha_7)\,, \\ \varpi_3 = e_1 + e_2 + \sqrt{2}(e_5 - e_7) = 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 6\alpha_4 + 4\alpha_5 + 4\alpha_6 + 2\alpha_7\,, \\ \varpi_4 = e_1 + \frac{1}{3\sqrt{2}}(4e_5 + e_6 - 5e_7) = 4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 9\alpha_4 + 6\alpha_5 + 6\alpha_6 + 3\alpha_7\,, \\ \varpi_5 = \sqrt{2}(e_5 + e_6) = \frac{1}{2}(6\alpha_1 + 12\alpha_2 + 18\alpha_3 + 15\alpha_4 + 10\alpha_5 + 9\alpha_6 + 5\alpha_7)\,, \\ \varpi_6 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4) + \frac{1}{\sqrt{2}}(e_5 - e_7) \\ = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7\,, \\ \varpi_7 = \frac{1}{\sqrt{2}}(e_5 + e_6 + e_7) = \frac{1}{2}(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 3\alpha_7)\,. \end{array} \tag{9.7}$$

It follows from (9.7) that

$$P(\mathbf{e_7}) = Q(\mathbf{e_7}) + \mathbb{Z}\varpi_7\,. \tag{9.8}$$

and the factor-group $P(\mathbf{e_7})/Q(\mathbf{e_7})$ is isomorphic μ_2.

The minimal root (9.5) defines the vertices of the fundamental alcove $C_{alc} = (0, \frac{1}{2}\varpi_1, \frac{1}{3}\varpi_2, \frac{1}{4}\varpi_3, \frac{1}{3}\varpi_4, \frac{1}{2}\varpi_5, \frac{1}{2}\varpi_6, \varpi_7)$. Then ϖ_7 generates λ_7

$$\lambda_7 = \left\{ \begin{array}{ll} \alpha_1 \to \alpha_5\,, & \\ \alpha_2 \to \alpha_4\,, & e_1 \to \frac{1}{2}(e_1+e_2+e_3-e_4)\,, \\ \alpha_3 \to \alpha_3\,, & e_2 \to \frac{1}{2}(e_1+e_2-e_3+e_4)\,, \\ \alpha_4 \to \alpha_2\,, & e_3 \to \frac{1}{2}(e_1-e_2+e_3+e_4)\,, \\ \alpha_5 \to \alpha_1\,, & e_4 \to \frac{1}{2}(-e_1+e_2+e_3+e_4)\,, \\ \alpha_6 \to \alpha_6\,, & e_5 \to -e_7\,, \\ \alpha_7 \to \alpha_0\,, & e_6 \to -e_6\,, \\ \alpha_0 \to \alpha_7\,, & e_7 \to -e_5\,. \end{array} \right. \tag{9.9}$$

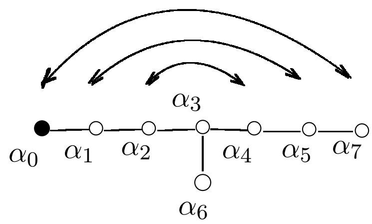


Fig.8 E_7, λ_7

Chevalley basis

The Lie algebra $\mathbf{e_7}$ can be defined in terms of $\mathbf{e_6}$ and its representations

$$\mathbf{e_7} = \mathbf{e_6} \oplus \mathcal{I}\,, \quad \mathcal{I} = \underline{27} \oplus \overline{\underline{27}} \oplus \underline{1}\,. \tag{9.10}$$

Here $\underline{27}$ and $\overline{\underline{27}}$ are two fundamental representations of $\mathbf{e_6}$ corresponding to the weights ϖ_1 and ϖ_5 (8.13). The scalar $\underline{1}$ corresponds to the generator $(e_5+e_6+e_7)$ in the Cartan subalgebra $\mathfrak{H}_{\mathbf{e_7}}$. It allows us to use in $\mathfrak{H}(\mathbf{e_7})$ the unrestricted canonical basis e_j, $j = 1,\dots,7$. It follows from (9.1) that the rest generators correspond to new 54 root subspaces. Similarly to (8.19) they are

$$\begin{array}{lll} \alpha^{(L,+)}_{(a,\pm)} = (\varpi^L_a \pm \frac{1}{\sqrt{2}}(e_5+e_7)) \to E^{L,+}_{a,\pm} & a = 1,\dots,8\,, \\ \alpha^{(R,+)}_{(a,\pm)} = (\varpi^R_a \pm \frac{1}{\sqrt{2}}(e_5+e_6)) \to E^{R,+}_{a,\pm}\,, & a = 1,\dots,8\,, \\ \alpha^{(V,+)}_{(a,\pm)} = (\varpi^V_a \pm \frac{1}{\sqrt{2}}(e_6+e_7)) \to E^{R,+}_{a,\pm}\,, & a = 1,\dots,8\,, \\ \pm\alpha^{(+)}_j = \pm\sqrt{2}e_j \to E_{j,\pm}\,, & j = 5,6,7\,. \end{array} \tag{9.11}$$

The positive root subspaces are $E^{A,+}_{a,+}$ and $E_{j,+}$.

The new generators are orthogonal to the $\mathbf{e_6}$ generators and (see (A.25.I))

$$(E^{A,+}_{a,+}, E^{B,+}_{b,-}) = \delta_{a,-b}\delta^{A,B}\,, \quad A,B = L,R,V\,, \tag{9.12}$$

$$(E_{j,\pm}, E_{k,\mp}) = \delta_{jk}\,.$$

GS basis

Since $\lambda_7^2 = 1$,

$$\mathbf{e_7} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \,, \quad \mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 + V \,. \tag{9.13}$$

It follows from (9.10) and Fig. 8 that $\mathfrak{g}_0 \sim \mathbf{e_6} + \underline{1}$ We will prove that $\tilde{\mathfrak{g}}_0 = \mathbf{f_4}$.

The root subsystem Π_1 that does not contain an orbit of λ passing through α_0 is $\Pi_{\mathbf{e_6}}$ (9.3). It follows from (9.9) that the λ_7-action on $\Pi_{\mathbf{e_6}}$ takes the form

$$\tilde{\lambda}_7 = \begin{cases} \alpha_1 \to \alpha_5 \,, \\ \alpha_2 \to \alpha_4 \,, \\ \alpha_3 \to \alpha_3 \,, \\ \alpha_4 \to \alpha_2 \,, \\ \alpha_5 \to \alpha_1 \,, \\ \alpha_6 \to \alpha_6 \,, \end{cases}$$

The set of orbits in $\Pi_1^\vee = \Pi_1$ is

$$\tilde{\Pi}^\vee = \Pi_1^\vee/\mu_2 = \Big(\tilde{\alpha}_1^\vee = \alpha_6 \,,\ \tilde{\alpha}_2^\vee = \alpha_3 \,,\ \tilde{\alpha}_3^\vee = \alpha_2 + \alpha_4 \,,\ \tilde{\alpha}_4^\vee = \alpha_1 + \alpha_5\Big) =$$

$$= \Big(e_4 + e_3 \,, e_2 - e_3 \,, e_1 - e_2 + e_3 - e_4 \,, \frac{1}{2}(e_4 - e_3 - e_2 - 3e_1) + \frac{1}{\sqrt{2}}(e_5 - e_7)\Big) \,. \tag{9.14}$$

It is the coroot basis in the invariant subalgebra $\tilde{\mathfrak{H}}_0$. The dual root system

$$\tilde{\alpha}_1 = e_3 + e_4 \,,\ \tilde{\alpha}_2 = e_2 - e_3 \,,\ \tilde{\alpha}_3 = \frac{1}{2}(e_1 - e_2 + e_3 - e_4) \,, \tag{9.15}$$

$$\tilde{\alpha}_4 = \frac{1}{4}(-3e_1 - e_2 - e_3 + e_4) + \frac{1}{2\sqrt{2}}(e_5 - e_7) \,.$$

defines simple roots of type $\mathbf{f_4}$. Then $\tilde{\mathfrak{H}}_0 = \mathfrak{H}(\mathbf{f_4})$.

GS basis in $\mathbf{e_6}$ subalgebra

Under the $\tilde{\lambda}_7$-action $\mathbf{e_6}$ is decomposed as

$$\mathbf{e_6} = \mathbf{f_4} \oplus \underline{26} \,, \tag{9.16}$$

where $\underline{26}$ is a fundamental representation of $\mathbf{f_4}$. In terms of (9.13) $\tilde{\mathfrak{g}}_0 = \mathbf{f_4}$ and $\underline{26} \subset \mathfrak{g}_1$.

Let us describe (9.16) in terms of the familiar decomposition (8.4)

$$\mathbf{e_6} = \mathbf{so(8)} \oplus \mathcal{J} = \underline{28} \oplus \underline{50} \tag{9.17}$$

taking into account the $\tilde{\lambda}_7$-action.

GS basis in so(8)-component.
The $\mathbf{so(8)}$ subalgebra is generated by the simple roots $(\alpha_2, \alpha_3, \alpha_4, \alpha_6)$. The $\tilde{\lambda}_7$ invariant subsystem $(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$ (9.15) forms B_3 root subsystem. It implies that

$$\mathbf{so(8)} = \mathbf{so(7)} \oplus \underline{7} , \quad \underline{28} = \underline{21} \oplus \underline{7} , \tag{9.18}$$

$$\tilde{\lambda}_7(\mathbf{so(7)}) = \mathbf{so(7)} \,,\ \tilde{\lambda}_7(\underline{7}) = -\underline{7} ,$$

where $\underline{7}$, is the fundamental representations of $\mathbf{so(7)}$.

The 24 root subspaces of $\mathbf{so(8)}$ contains 18 root subspaces of $\mathbf{so(7)}$. Then according with the definition of $\mathbf{so(8)}$ root subspaces (8.18) we define the GS-basis in $\mathbf{so(8)}$.

$\mathbf{so(7)}$	$\underline{7}$
$E^0_{12} = E_{12} + E_{34}$	$\mathfrak{t}^1_{12} = \frac{1}{\sqrt{2}}(E_{12} - E_{34})$
$E^0_{13} = E_{13} + E_{24}$	$\mathfrak{t}^1_{13} = \frac{1}{\sqrt{2}}(E_{13} - E_{24})$
$E^0_{15} = E_{15} + E_{26}$	$\mathfrak{t}^1_{15} = \frac{1}{\sqrt{2}}(E_{15} - E_{26})$
$E^0_{23} = E_{23}\,,\ E^0_{14} = E_{14}\,,$	$\mathfrak{t}^1_{j,1}\,,\ j = 2,3,5$
$E^0_{35} = E_{35}\,,\ E^0_{25} = E_{25}$	$\mathfrak{h}^1_{\alpha_2}$ (9.20)
$E^0_{16} = E_{16}\,,\ E^0_{17} = E_{17}$	

Table 2. GS basis in $\mathbf{so(8)}$.

The left column represents 9 positive root subspaces of $\mathbf{so(7)}$, and the right column represents the root subspaces and a Cartan element in $\underline{7}$. The $\mathbf{so(8)}$ roots that parameterized the GS-basis are

$$\begin{array}{ll} \alpha_{12} = (e_1 - e_2) \to \mathfrak{t}^1_{12}\,, & -\alpha_{12} = (e_2 - e_1) \to \mathfrak{t}^1_{21}\,, \\ \alpha_{13} = (e_1 - e_3) \to \mathfrak{t}^1_{13}\,, & -\alpha_{13} = (e_3 - e_1) \to \mathfrak{t}^1_{31}\,, \\ \alpha_{15} = (e_1 + e_4) \to \mathfrak{t}^1_{15}\,, & -\alpha_{15} = (-e_1 - e_4) \to \mathfrak{t}^1_{51}\,, \end{array} \tag{9.19}$$

Consider the embedding of $\mathfrak{H}(\mathbf{so(7)})$ in $\mathfrak{H}(\mathbf{so(8)})$. Remember that the basis in $\mathfrak{H}(\mathbf{so(7)})$ is $(\tilde{\alpha}^\vee_k\,,\ k = 1,2,3)$. In addition we have the anti-invariant generator

$$\mathfrak{h}^1_{\alpha_2} = \frac{1}{\sqrt{2}}(\alpha_2 - \alpha_4) = \frac{1}{\sqrt{2}}(-e_1 + e_2 + e_3 - e_4)\,, \tag{9.20}$$

$$\mathfrak{H}(\mathbf{so(8)}) = \mathfrak{H}(\mathbf{so(7)}) \oplus \mathbb{C}\mathfrak{h}^1_{\alpha_2}\,.$$

In this way we have defined the $\tilde{\lambda}_7$-action on $\mathbf{so(8)}$ component in (9.17). In correspondence with (9.16) we have

$$\mathbf{so(7)} \subset \mathbf{f_4}\,,\quad \underline{7} \subset \underline{26}\,.$$

GS basis in $\mathcal{J}$-component

Now consider the λ-action on the space $\mathcal{J}$ (9.17). It is represented by 48 root subspaces (8.3) and two elements from $\mathfrak{H}(\mathbf{e_6})$. Let us define the latter generators. They are

$$\mathfrak{h}^1_{\alpha_1} = \frac{1}{\sqrt{2}}(\alpha_1 - \alpha_5) = \frac{1}{\sqrt{2}}\Big(\frac{1}{2}(e_1 - e_2 - e_3 + e_4) + \frac{1}{\sqrt{2}}(e_5 - 2e_6 + e_7)\Big) \tag{9.21}$$

and the already defined invariant generator $\tilde{\alpha}^\vee_4 = 2\tilde{\alpha}_4$ (9.15) that completes $\mathfrak{H}(\mathbf{so(7)})$ to $\mathfrak{H}(\mathbf{f_4})$.

The $\tilde{\lambda}_7$-action on the the root subspaces takes the form

$$\tilde{\lambda}_7\ :\ \begin{cases} (P^R \pm \frac{1}{\sqrt{2}}(e_5 - e_6)) \leftrightarrow (P^V \pm \frac{1}{\sqrt{2}}(e_6 - e_7))\,, \\ (P^L \pm \frac{1}{\sqrt{2}}(e_5 - e_7)) \leftrightarrow (P^L \pm \frac{1}{\sqrt{2}}(e_5 - e_7))\,. \end{cases}$$

Since $\tilde{\lambda}_7\,:\,\varpi^L_4 \to -\varpi^L_4 = \varpi^L_8$, all roots $P^L \pm \frac{1}{\sqrt{2}}(e_5 - e_7)$ are fixed under the $\tilde{\lambda}_7$-action, except $(\varpi^L_4 = \frac{1}{2}(e_1 - e_2 - e_3 + e_4)) \pm \frac{1}{\sqrt{2}}(e_5 - e_7)$. Then from the $\tilde{\lambda}_7$ action on the Chevalley basis in $\mathbf{e_6}$ (8.19) we obtain generators of the GS basis

$$\begin{array}{rl} \mathfrak{t}^{R,k}_{a,\pm} = \frac{1}{\sqrt{2}}(E^R_{a,\pm} + (-1)^k E^V_{a,\pm})\,, & k = 0,1\,, \\ \mathfrak{t}^{L,k}_{4,\pm} = \frac{1}{\sqrt{2}}(E^L_{4,\pm} + (-1)^k E^L_{8,\pm})\,, & k = 0,1\,, \\ \mathfrak{t}^{L,0}_{a,\pm} = E^L_{a,\pm}\,, & a \neq 4,8\,. \end{array} \tag{9.22}$$

The μ_2-gradation $\mathcal{J} = \mathcal{J}_0 \oplus \mathcal{J}_1$ takes the form

$$\mathcal{J}_0 = \{\sqrt{2}\mathfrak{t}^{R,0}_{a,\pm} = E_{\tilde{\alpha}^R_{a,\pm}}\,,\ a = 1,\ldots,8\,,\ \ \sqrt{2}\mathfrak{t}^{L,0}_{a,\pm} = E_{\tilde{\alpha}^L_{a,\pm}}\,,\ a \neq 4,8\,,\ \ \sqrt{2}\mathfrak{t}^{L,0}_{4,\pm} = E_{\tilde{\alpha}^L_{4,\pm}}\,,\ \tilde{\alpha}^\vee_4\}\,, \tag{9.23}$$

$$\mathcal{J}_1 = \{\mathfrak{t}^{R,1}_{a,\pm}\,,\ a = 1,\ldots,8\,,\ \ \mathfrak{t}^{L,1}_{4,\pm}\,,\ \mathfrak{h}^1_{\alpha_1}\}\,, \tag{9.24}$$

$$\dim\,\mathcal{J}_0 = 31\,,\quad \dim\,\mathcal{J}_1 = 19\,.$$

Here $E_{\tilde{\alpha}^R_{a,\pm}}$ are invariant root subspaces constructed from the root subspaces of $\mathbf{e_6}$ (8.19). Finally, by comparing (9.16) and (9.17) we come to the decompositions

$$\mathbf{f_4} = \mathbf{so(7)} \oplus \mathcal{J}_0\,,\quad (52 = 21 + 31)\,, \tag{9.25}$$

$$\underline{26} = \mathcal{J}_1 \oplus \underline{7}\,,\quad (26 = 19 + 7)\,, \tag{9.26}$$

where $\underline{7}$ is represented by the right column in Table 2.

Chevalley basis in $\mathbf{f_4}$

In what follows we need the Chevalley basis in $\mathbf{f_4}$ (9.25) in terms of the Chevalley basis in $\mathbf{e_6}$. We pass to the canonical basis in $\mathfrak{H}(\mathbf{f_4})$. From (9.14) we find

$$\mathfrak{H}(\mathbf{f_4}) = \{\tilde{\mathbf{u}} = u_1e_1 + u_2e_2 + u_3e_3 + u_4e_4 + u_5e_5 - u_5e_7\}\,, \tag{9.27}$$

where

$$u_1 - u_2 - u_3 + u_4 = 0\,. \tag{9.28}$$

In other terms it can be written in the form (see (9.25))

$$\mathfrak{H}(\mathbf{f_4}) = \mathfrak{H}(\mathbf{so(7)}) \oplus \mathbb{C}(\frac{1}{2}(-3e_1 - e_2 - e_3 + e_4) + \frac{1}{\sqrt{2}}(e_5 - e_7))\,. \tag{9.29}$$

It follows from (9.23), (9.25) that the $\mathbf{f_4}$ roots are

$$R(\mathbf{f_4}) = R(\mathbf{so(7)}) \cup R(\mathcal{J}_0)\,, \tag{9.30}$$

$$R^+(\mathcal{J}_0) = \{\tilde{\alpha}^R_{a,+} = \frac{1}{2}(\varpi^R_a + \varpi^V_a + \frac{1}{\sqrt{2}}(e_5 - e_7))\,,\ \tilde{\alpha}^L_{a,+} = \frac{1}{2}(\varpi^L_a + \frac{1}{\sqrt{2}}(e_5 - e_7))\,,\ (a \neq 4,8)\,,\ \}\,,$$

$$\tilde{\alpha}_4\,,\ (9.15)\,,\quad R^+(\mathbf{so(7)}) = \Big\{\tilde{\alpha}_j\,,\ j = 1,2,3,\ (9.15)\,,$$

$$\frac{1}{2}(e_1 + e_2 - e_3 - e_4)\,,\ \frac{1}{2}(e_1 + e_2 + e_3 + e_4)\,,\ (e_1 - e_4)\,,\ (e_2 + e_4)\,,\ (e_1 + e_3)\,,\ (e_1 + e_2)\Big\}\,.$$

The $\mathbf{so(7)}$ root subspaces are read of from Table 2, while the root subspaces in $\mathcal{J}_0$ are defined in (9.23).

On $\mathfrak{H}(\mathbf{f_4})$ with the basis $H_{\tilde{\alpha}_j} = \tilde{\alpha}^\vee_j$, $j = 1,\ldots,4$ (9.14) the Killing form is defined by the $\mathbf{f_4}$ Cartan matrix (A.24.I). On $\mathfrak{L}(\mathbf{f_4})$ the Killing form is is

$$\begin{gathered}(E^0_{23}, E^0_{32}) = (E^0_{14}, E^0_{41}) = (E^0_{35}, E^0_{53}) = (E^0_{25}, E^0_{52}) = (E^0_{16}, E^0_{61}) = (E^0_{17}, E^0_{71}) = 1\,,\\ (E^0_{12}, E^0_{12}) = (E^0_{13}, E^0_{31}) = (E^0_{15}, E^0_{51}) = 2\,,\\ (\mathfrak{t}^{R,0}_{a,+}, \mathfrak{t}^{R,0}_{b,-}) = 2\delta_{a,-b}\,,\quad (\mathfrak{t}^{L,0}_{a,+}, \mathfrak{t}^{L,0}_{b,-}) = 2\delta_{a,-b}\,.\end{gathered} \tag{9.31}$$

GS basis in $\mathcal{I}$

Consider the $\tilde{\lambda}_7$-action on the subspace $\mathcal{I}$ in (9.10). 55 Chevalley generators in $\mathcal{I}$ (9.11) form the GS basis similarly to (9.22). Then we come to the GS basis

$$\begin{array}{l}\mathfrak{t}^{R,+,k}_{a,+} = \frac{1}{\sqrt{2}}(E^{R,+}_{a,+} + (-1)^k E^{V,+}_{a,-})\,, \quad k=0,1\,,\\ \mathfrak{t}^{R,+,k}_{a,-} = \frac{1}{\sqrt{2}}(E^{R,+}_{a,-} + (-1)^k E^{V,+}_{a,+})\,, \quad k=0,1\,,\\ \mathfrak{t}^{L,+,k}_{4,+} = \frac{1}{\sqrt{2}}(E^{L,+}_{4,+} + (-1)^k E^{L,+}_{8,-})\,, \quad k=0,1\,,\\ \mathfrak{t}^{L,+,k}_{4,-} = \frac{1}{\sqrt{2}}(E^{L,+}_{4,-} + (-1)^k E^{L,+}_{8,+})\,, \quad k=0,1\,,\\ \mathfrak{t}^{L,+,k}_{a,+} = \frac{1}{\sqrt{2}}(E^{L,+}_{a,+} + (-1)^k E^{L,+}_{a,-})\,, \quad a\neq 4,8\,, \quad k=0,1\,,\\ \mathfrak{t}^{k}_{5,\pm} = \frac{1}{\sqrt{2}}(E_{5,\pm} + (-1)^k E_{7,\mp})\,, \quad k=0,1\,,\\ \mathfrak{t}^{k}_{6,+} = \frac{1}{\sqrt{2}}(E_{6,+} + (-1)^k E_{6,-}) \quad k=0,1\,,\\ \mathfrak{h}^1_{e_5} = \frac{1}{\sqrt{3}}(e_5+e_6+e_7)\,.\end{array} \tag{9.32}$$

The generators with $k=0$ form the GS basis in V and with $k=1$ along with the GS basis in $\underline{26}$ (9.26) form basis in $\mathfrak{g}_1$.

Thus, we come to the following GS basis

$$\mathbf{e_7} = \mathbf{f_4} \oplus V \oplus \mathfrak{g}_1\,, \quad (\underline{133} = \underline{52} \oplus \underline{27} \oplus \underline{54})\,,$$

$$V = \{\mathfrak{t}^{R,+,0}_{a,\pm}\,,\ \mathfrak{t}^{L,+,0}_{a,+}\,, (a\neq 4,8)\,,\ \mathfrak{t}^{L,+,0}_{4,\pm}\,,\ \mathfrak{t}^{0}_{5,\pm}\,,\ \mathfrak{t}^{0}_{6,+}\}\,, \tag{9.33}$$

$$\mathfrak{g}_1 = \Big\{\mathfrak{t}^{R,1}_{a,\pm}\,,\ \mathfrak{t}^{L,1}_{4,\pm}\,,\ \mathfrak{t}^{R,+,1}_{a,\pm}\,,\ \mathfrak{t}^{L,+,1}_{a,+}\,,\ (a\neq 4,8)\,,\ \mathfrak{t}^{L,+,1}_{4,\pm}\,,$$

$$\mathfrak{t}^1_{1j}\,,\ \mathfrak{t}^1_{j1}\,, (j=2,3,5)\,,\ \mathfrak{t}^1_{5,\pm}\,,\ \mathfrak{t}^1_{6,+}\,,\ \mathfrak{h}^1_{\alpha_1}\,,\ \mathfrak{h}^1_{\alpha_2}\,,\ \mathfrak{h}^1_{e_5}\Big\}\,.$$

The Killing form on the GS generators takes the form

$$\begin{array}{c}(\mathfrak{t}^{R,+,k_1}_{a,+}, \mathfrak{t}^{R,+,k_2}_{b,-}) = \delta_{a,-b}\delta^{k_1+k_2,0\,,\ mod(2)}\,, \quad (\mathfrak{t}^{L,+,k_1}_{4,\pm}, \mathfrak{t}^{L,+,k_2}_{4,\pm}) = \delta^{k_1+k_2,0\,,\ mod(2)}\,,\\ (\mathfrak{t}^{L,k_1}_{4,\pm}, \mathfrak{t}^{L,k_2}_{4,\mp}) = \delta^{k_1+k_2,0\,,\ mod(2)}\,, \quad (\mathfrak{t}^{R,k_1}_{a,\pm}, \mathfrak{t}^{R,k_2}_{b,\mp}) = \delta_{a,-b}\delta^{k_1+k_2,0\,,\ mod(2)}\,,\\ (\mathfrak{t}^{L,+,k_1}_{a,+}, \mathfrak{t}^{L,+,k_2}_{b,+}) = \delta_{a,-b}\delta^{k_1+k_2,0\,,\ mod(2)}\,, \ (a\neq 4,8)\,, \quad (t^1_{1,j}, t^1_{k,1}) = \delta_{jk}\,,\\ (\mathfrak{h}^1_{e_6}, \mathfrak{h}^1_{e_6}) = 1\,, \quad (\mathfrak{t}^{k_1}_{5,+}, \mathfrak{t}^{k_2}_{5,-}) = \delta^{k_1+k_2,0\,,\ mod(2)}\,, \quad (\mathfrak{t}^{k_1}_{6,+}, \mathfrak{t}^{k_2}_{6,-}) = \delta^{k_1+k_2,0\,,\ mod(2)}\,.\end{array} \tag{9.34}$$

It allows us to define the dual basis (5.9.I)

$$\begin{array}{c}\mathfrak{T}^{R,+,k}_{a,+} = \mathfrak{t}^{R,+,k}_{-a,-}\,,\ \ \mathfrak{T}^{R,+,k}_{a,-} = \mathfrak{t}^{R,+,k}_{-a,+}\,,\ \ \mathfrak{T}^{L,+,k}_{4,\pm} = \mathfrak{t}^{L,+,k}_{4,\pm}\,,\\ \mathfrak{T}^{R,k}_{a,+} = \mathfrak{t}^{R,k}_{-a,-}\,,\ \ \mathfrak{T}^{L,+,k}_{4,\pm} = \mathfrak{t}^{L,+,k}_{4,\mp}\,,\\ \mathfrak{T}^{L,+,k}_{a,\pm} = \mathfrak{t}^{L,+,k}_{-a,\mp}\,,\ (a\neq 4,8)\,,\ \ \mathfrak{H}^1_{e_5} = \mathfrak{h}^1_{e_5} = \dfrac{1}{\sqrt{3}}(e_5+e_6+e_7)\,,\\ \mathfrak{T}^{k}_{5,+} = \mathfrak{t}^{k}_{5,-}\,,\quad \mathfrak{T}^{k}_{6,+} = \mathfrak{t}^{k}_{6,-}\,,\quad \mathfrak{T}^1_{1,j} = t^1_{j,1}\,,\quad \mathfrak{T}^1_{j,1} = t^1_{1,j}\,.\end{array} \tag{9.35}$$

It follows from (9.20) and (9.21) that the scalar product matrix $a_{j,k} = (\mathfrak{h}^1_{\alpha_j}, \mathfrak{h}^1_{\alpha_k})$ for $(j,k=1,2)$ is the Cartan matrix A_2. We need the inverse matrix to define the dual basis (5.15.I)

$$\mathcal{A} = \frac{1}{3}\begin{pmatrix} 2 & 1\\ 1 & 2\end{pmatrix}\,. \tag{9.36}$$

Thus,

$$\mathfrak{H}^1_{\alpha_1} = \frac{2}{3}\mathfrak{h}^1_{\alpha_1} + \frac{1}{3}\mathfrak{h}^1_{\alpha_2} = \frac{1}{3}(e_5 - 2e_6 + e_7)\,, \tag{9.37}$$

$$\mathfrak{H}^1_{\alpha_2} = \frac{1}{3}\mathfrak{h}^1_{\alpha_1} + \frac{2}{3}\mathfrak{h}^1_{\alpha_2} = \frac{1}{2\sqrt{2}}(-e_1 + e_2 - e_3 - e_4) + \frac{1}{6}(e_5 - 2e_6 + e_7)\,. \tag{9.38}$$

The scalar product $(\mathfrak{H}^1_{\alpha_j}, \mathfrak{H}^1_{\alpha_k})$ is defined by $\mathcal{A}$ (9.36), while $\mathfrak{H}^1_{e_5}$ is orthogonal to them.

Lax operators and Hamiltonians

Trivial bundles

The moduli space of trivial $\bar{G} = E_7$ bundles is the quotient

$$\mathfrak{H}(\mathbf{e_7})/W_{\mathbf{e_7}} \ltimes (\tau(Q(\mathbf{e_7}) + Q(\mathbf{e_7}))\,.$$

The moduli space of trivial $G^{ad} = E_7/\mu_2$ bundles is the quotient

$$\mathfrak{H}(\mathbf{e_7})/W_{\mathbf{e_7}} \ltimes (\tau(P(\mathbf{e_7}) + P(\mathbf{e_7}))\,,$$

where $P(\mathbf{e_7})$ are defined by the basis (9.8).

For trivial bundles Lax operator takes the form

$$L^{CM}_{\mathbf{e_7}}(z) = L^{CM}_{\mathbf{e_6}}(z) + \sum_{a=1}^{8}\sum_{A=L,R,V} S^{A,+}_{a,\pm}\phi((\mathbf{u},\varpi^A_a) \pm \frac{1}{\sqrt{2}}(u_j + u_k), z)E^{A,+}_{a,\pm} + \sum_{j=5,6,7} S_{j,\pm}\phi(\pm\sqrt{2}u_j, z)E_{j,\pm}\,.$$

Here we have the correspondence $L \to (j = 5,7)$, $R \to (j = 5,6)$, $V \to (j = 6,7)$. In contrast with $\mathbf{e_6}$ there is no restrictions $\sum_{k=1}^{7} v_k = \sum_{k=1}^{7} u_k = 0$.

The Hamiltonian after the symplectic reduction with respect to the $\mathcal{H}(\mathbf{e_7})$ action assumes the form

$$H^{CM}_{\mathbf{e_7}} = H^{CM}_{\mathbf{e_6}} + \sum_{a=1}^{4}\sum_{A=L,R,V} S^{A,+}_{a,+}S^{A,+}_{a,-}E_2((\mathbf{u},\varpi^A_a) + \frac{1}{\sqrt{2}}(u_j + u_k)) + \sum_{j=5,6,7} S_{j,+}S_{j,-}E_2(\sqrt{2}u_j, z)\,.$$

Nontrivial bundles

The moduli space.
The moduli space are described by elements $\tilde{\mathbf{u}} \in \mathfrak{H}(\mathbf{f_4})$ (9.27). The Weyl group $W(\mathbf{f_4})$ is generated by reflections with respect the planes, orthogonal to simple roots (9.15).

The coroot lattice $\tilde{Q}^\vee(\mathbf{f_4})$ is generated by the simple coroots (9.14). The moduli space of nontrivial E_7 bundles is defined as

$$\mathfrak{H}(\mathbf{f_4})/(W(\mathbf{f_4}) \ltimes (\tau\tilde{Q}^\vee(\mathbf{f_4}) + \tilde{Q}^\vee(\mathbf{f_4})))\,.$$

F_4 **Calogero-Moser system.**
Represent the Lax operator in the form (6.7.I)

$$L(z) = \tilde{L}_0(z) + L'(z) + L_1(z)\,,$$

where $\tilde{L}_0(z) = L^{CM}_{\mathbf{f_4}}(z)$. In defined below expression $\tilde{\mathbf{v}} = (v_1, v_2, v_3, v_4, v_5)$ the momentum vector satisfies the same restriction as the vector $\tilde{\mathbf{u}}$ (9.28). It follows from (9.25) it takes the form

$$L^{CM}_{\mathbf{f_4}}(z) = L^{CM}_{\mathbf{so(7)}}(z) + \frac{1}{4}\Big(-3(v_1 + S_{0,1}E_1(z))e_1 - (v_2 + S_{0,2}E_1(z))e_2 - (v_3 + S_{0,3}E_1(z))e_3$$

$$+(v_4 + S_{0,4}E_1(z))e_4) + \frac{1}{2\sqrt{2}}((v_5 + S_{0,5}E_1(z))e_5 - (v_5 + S_{0,5}E_1(z))e_7)\Big) +$$

$$+\sum_{a=1}^{8}\Big(S^{R,0}_{a,+}\phi((\tilde{\mathbf{u}}, \varpi^R_a) + \frac{1}{\sqrt{2}}u_5, z)\mathfrak{t}^{R,0}_{-a,-} + S^{R,0}_{-a,-}\phi(-(\tilde{\mathbf{u}}, \varpi^R_a) - \frac{1}{\sqrt{2}}u_5, z)\mathfrak{t}^{R,0}_{a,+}\Big)$$

$$S^{L,0}_{4,+}\phi((\tilde{\mathbf{u}}, \varpi^L_4) + \sqrt{2}u_5, z)\mathfrak{t}^{L,0}_{4,-} + S^{L,0}_{4,-}\phi(-(\tilde{\mathbf{u}}, \varpi^L_4) - \sqrt{2}u_5, z)\mathfrak{t}^{L,0}_{4,+} +$$

$$\sum_{a\neq 4}^{8}\Big(S^{L,0}_{a,+}\phi((\tilde{\mathbf{u}}, \varpi^L_a) + \sqrt{2}u_5, z)\mathfrak{t}^{L,0}_{-a,-} + S^{L,0}_{-a,-}\phi(-(\tilde{\mathbf{u}}, \varpi^L_a) - \sqrt{2}u_5, z)\mathfrak{t}^{R,0}_{a,+}\Big)\,.$$

We find the corresponding quadratic Hamiltonian

$$H^{CM}_{\mathbf{f_4}} = H^{CM}_{\mathbf{so(7)}} + \frac{1}{2}\Big(v_1^2 + v_2^2 + v_3^2 + v_4^2 + 2v_5^2\Big)$$

$$+\sum_{a=1}^{8} S^{R,0}_{a,+}S^{R,0}_{-a,-}E_2((\tilde{\mathbf{u}}, \varpi^R_a) + \frac{1}{\sqrt{2}}u_5) + S^{L,0}_{4,+}S^{L,0}_{4,-}E_2((\tilde{\mathbf{u}}, \varpi^L_4) + \sqrt{2}u_5)$$

$$+\sum_{a\neq 4}^{8} S^{L,0}_{a,+}S^{L,0}_{-a,-}E_2((\tilde{\mathbf{u}}, \varpi^L_a) + \sqrt{2}u_5)\,.$$

The Lax operators and Hamiltonians

Define as in the general case (6.14.I) the function

$$\varphi^k_\beta(\tilde{\mathbf{u}}, z) = \mathbf{e}\left(\langle\kappa_{\mathbf{e_7}}, \beta\rangle z\right)\phi(\langle\kappa_{\mathbf{e_7}} - \tilde{\mathbf{u}}, \beta\rangle + k/2, z)\,, \quad (k = 0, 1)\,,$$

where $\kappa_{\mathbf{e_7}}$ (9.6) and $\beta \in R(\mathbf{e_7})$ generating the GS basis. The following $R(\mathbf{e_7})$ roots define GS generators (9.33)

$$\beta = \Big\{\alpha^{(L,+)}_{(a,\pm)}\,, \quad \alpha^{(R,+)}_{(a,\pm)}\,, \quad \alpha_{1j} = e_1 - e_j\,,\ (j = 2, 3, 5)\,, \quad (9.11)\,, \qquad (9.39)$$

$$\alpha^L_{(a,\pm)}\,,\ \alpha^R_{(a,\pm)}\ (8.19)\,, \quad \pm\,\alpha^{(+)}_j,\ (j = 5, 6, 7)\,,\ (9.19)\,, \quad \Big\}\,.$$

Then following (6.15.I) and (6.17.I) from (9.33) we find

$$L'_0(z) = \sum_{a=1}^{8} S^{R,+,0}_{a,\pm}\varphi^0_{\alpha^{(L,+)}_{(a,\pm)}}(\tilde{\mathbf{u}}, z)\mathfrak{t}^{R,+,0}_{-a,\mp} + \sum_{a\neq 4,8} S^{L,+,0}_{a,\pm}\varphi^0_{\alpha^{(L,+)}_{(a,\pm)}}(\tilde{\mathbf{u}}, z)\mathfrak{t}^{L,+,0}_{-a,\mp} +$$

$$S^{L,+,0}_{4,\pm}\varphi^0_{\alpha^{(L,+)}_{(4,\pm)}}(\tilde{\mathbf{u}}, z)\mathfrak{t}^{L,+,0}_{4,\pm} + S^0_{5,\pm}\varphi^0_{\pm\alpha^{(+)}_{(5)}}(\tilde{\mathbf{u}}, z)\mathfrak{t}^0_{5,\mp} + S^0_{6,\pm}\varphi^0_{\pm\alpha^{(+)}_{(6)}}(\tilde{\mathbf{u}}, z)\mathfrak{t}^0_{6,\mp}\,.$$

$$L_1(z) = \Big(S^1_{\alpha_1}\mathfrak{H}^1_{\alpha_1} + S^1_{\alpha_2}\mathfrak{H}^1_{\alpha_2} + S^1_{e_5}\mathfrak{h}^1_{e_5}\Big)\phi(\frac{1}{2}, z)$$

$$\sum_{a=1}^{8} S^{R,1}_{a,\pm}\varphi^1_{\alpha^{(R)}_{(a,\pm)}}(\tilde{\mathbf{u}},z)\mathbf{t}^{R,1}_{a,\mp} + S^{L,1}_{4,\pm}\varphi^1_{\alpha^{(L)}_{(4,\pm)}}(\tilde{\mathbf{u}},z)\mathbf{t}^{L,1}_{4,\mp}$$

$$+\sum_{a=1}^{8} S^{R,+,1}_{a,\pm}\varphi^1_{\alpha^{(R,+)}_{(a,\pm)}}(\tilde{\mathbf{u}},z)\mathbf{t}^{R,+,1}_{-a,\mp} + \sum_{a\neq 4,8} S^{L,+,1}_{a,\pm}\varphi^1_{\alpha^{(L,+)}_{(a,\pm)}}(\tilde{\mathbf{u}},z)\mathbf{t}^{L,+,1}_{-a,\mp} +$$

$$\sum_{j=2,3,4,5}\Big(S^1_{1,j}\varphi^1_{\alpha_{(1,j)}}(\tilde{\mathbf{u}},z)t^1_{j,1} + S^1_{j,1}\varphi^1_{\alpha_{(j,1)}}(\tilde{\mathbf{u}},z)t^1_{1,j}\Big) +$$

$$S^{L,+,1}_{4,\pm}\varphi^1_{\alpha^{(L,+)}_{(4,\pm)}}(\tilde{\mathbf{u}},z)\mathbf{t}^{L,+1}_{4,\mp} + S^1_{5,\pm}\varphi^1_{\pm\alpha^{(+)}_{(5)}}(\tilde{\mathbf{u}},z)\mathbf{t}^1_{5,\mp} + S^1_{6,\pm}\varphi^1_{\pm\alpha^{(+)}_{(6)}}(\tilde{\mathbf{u}},z)\mathbf{t}^1_{6,\mp}\,.$$

where for $\mathfrak{H}^1_{\alpha_1}$ and $\mathfrak{H}^1_{\alpha_2}$ see (9.37) and (9.38).

For the E_7 quadratic Hamiltonians we have

$$H_{\mathbf{e_7}} = H^{CM}_{\mathbf{f_4}} + H'_0 + H_1\,,$$

where H'_0 comes from $\frac{1}{2}(L_0^{'2})$ and H_1 from $\frac{1}{2}(L_1^2)$. To calculate the Hamiltonians we use (9.34) and (9.36) for scalar products of the dual Cartan generators $\mathfrak{H}^1_{\alpha_j}$. Then

$$-H'_0 = \sum_{a=1}^{8} S^{R,+,0}_{a,+}S^{R,+,0}_{-a,-}E_2(\langle\alpha^{(L,+)}_{(a,+)},\tilde{\mathbf{u}}\rangle) + \sum_{a\neq 4,8} S^{L,+,0}_{a,+}S^{L,+,0}_{-a,-}E_2(\langle\alpha^{(L,+)}_{(a,+)},\tilde{\mathbf{u}}\rangle) +$$

$$S^{L,+,0}_{4,+}S^{L,+,0}_{4,-}E_2(\langle\alpha^{(L,+)}_{(4,+)},\tilde{\mathbf{u}}\rangle) + S^0_{5,+}S^0_{5,-}E_2(\langle\alpha^{(+)}_{(5)},\tilde{\mathbf{u}}\rangle) + S^0_{6,+}S^0_{6,-}E_2(\langle\alpha^{(+)}_{(6)},\tilde{\mathbf{u}}\rangle)\,.$$

$$H_1 = \frac{2}{3}\Big((S^1_{\alpha_1})^2 + (S^1_{\alpha_2})^2 + S^1_{\alpha_1}S^1_{\alpha_2}\Big)E_2\Big(\frac{1}{2}\Big) + (S^1_{e_5})^2E_2\Big(\frac{1}{2}\Big)$$

$$+\sum_{a=1}^{8} S^{R,1}_{a,+}S^{R,1}_{-a,-}E_2(\langle\alpha^{(R)}_{(a,+)},\tilde{\mathbf{u}}+\frac{1}{2}\rangle) + S^{L,1}_{4,+}S^{L,1}_{4,-}E_2(\langle\alpha^{(L)}_{(4,+)},\tilde{\mathbf{u}}\rangle + \frac{1}{2})$$

$$+\sum_{a=1}^{8} S^{R,+,1}_{a,+}S^{R,+,1}_{-a,-}E_2(\langle\alpha^{(R,+)}_{(a,+},\tilde{\mathbf{u}}\rangle + \frac{1}{2}) + \sum_{a\neq 4,8} S^{L,+,1}_{a,+}S^{L,+,1}_{-a,-}E_2(\langle\alpha^{(L,+)}_{(a,+)},\tilde{\mathbf{u}}\rangle + \frac{1}{2})$$

$$+\sum_{j=2,3,4,5} S^1_{1,j}S^1_{j,1}E_2(\langle\alpha_{(1,j)},\tilde{\mathbf{u}}\rangle + \frac{1}{2}) + S^{L,+,1}_{4,+}S^{L,+,1}_{4,-}E_2(\langle\alpha^{(L,+)}_{(4,+)},\tilde{\mathbf{u}}\rangle + \frac{1}{2})$$

$$+S^1_{5,+}S^1_{5,-}E_2(\langle\alpha^{(+)}_{(5)},\tilde{\mathbf{u}}\rangle + \frac{1}{2}) + S^1_{6,+}S^1_{6,-}E_2(\langle\alpha^{(+)}_{(6)},\tilde{\mathbf{u}}\rangle + \frac{1}{2})\,,$$

where $\tilde{\mathbf{u}}$ is defined by (9.27), (9.28).

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